# Calculus I - MAC 2311 - Section 001

## Quiz 7 - Solutions

03/28/2018

1) a) [1.5 points] Give the definition of a critical number of a function $f$.

*A* critical number *of a function $f$ is a number $c$ in the domain of $f$ such that either $f'(c) = 0$ or $f'(c)$ does not exist.*

b) [1.5 points] State the Mean Value Theorem.

*Let $f$ be a function which is continuous on the closed interval $[a, b]$ and differentiable on the open interval $(a, b)$. Then there exists $c$ in $(a, b)$ such that*

$$f'(c) = \frac{f(b) - f(a)}{b - a}.$$

2) [4 points] Find the absolute maximum and minimum values of the function

$$f(x) = x^2 e^{-x}$$

on the closed interval $[1, 3]$.

*Solution:*

Since $f$ is a continuous function on the closed interval $[1, 3]$, the Extreme Value Theorem guarantees that $f$ attains an absolute maximum value and an absolute minimum value on $[1, 3]$. Let us find them!

- *Compute the values of $f$ at the endpoints of the interval $[1, 3]$.*

  We have $f(1) = 1 \cdot e^{-1} = \frac{1}{e} \sim 0.36$ and $f(3) = 3^2 e^{-3} = \frac{9}{e^3} \sim 0.44$.

- *Find the critical numbers of $f$ in $(1, 3)$ and their corresponding values.*

  Since $f$ is differentiable everywhere, its critical numbers in $(1, 3)$ are all the numbers $c$ in $(1, 3)$ such that $f'(c) = 0$.
  Here we have:

  $$f'(x) = (x^2 e^{-x})' = 2xe^{-x} + x^2 e^{-x} \cdot (-1) = 2xe^{-x} - x^2 e^{-x} = xe^{-x}(2 - x).$$

  Thus $f'(x) = 0$ if and only if $x = 0$ or $x = 2$ (note that $e^{-x} \neq 0$ for all $x$). Now only 2 is inside the interval $(1, 3)$ and the corresponding value is $f(2) = 2^2 e^{-2} = \frac{4}{e^2} \sim 0.54$.

- *Compare the values obtained in step 1 and step 2 and return the absolute maximum and the absolute minimum values of $f$.*

  We have $f(1) \sim 0.36 < f(3) \sim 0.44 < f(2) \sim 0.54$, so the absolute maximum value of $f$ on $[1, 3]$ is $f(2)$ and the absolute minimum value of $f$ on $[1, 3]$ is $f(2)$.

3) [4 points] Let $f$ be a differentiable function such that $f'(x) \leq 2$ for all $x$ in $\mathbb{R}$. If $f(0) = 3$, what is the greatest value that $f$ may attain at 2?

*Solution:*

We consider the function $f$ on the closed interval $[0, 2]$. Since $f$ is a function which is differentiable everywhere, we have that in particular $f$ is continuous on $[0, 2]$ and differentiable on $(0,2)$. Thus, by the Mean Value Theorem, there exists $c$ in $(0,2)$ such that

$$f'(c) = \frac{f(2)-f(0)}{2-0} \overset{f(0)=3}{=} \frac{f(2)-3}{2}.$$

By hypothesis $f'(c) \leq 2$. Therefore we have:

$$\frac{f(2)-3}{2} \leq 2 \Leftrightarrow f(2)-3 \leq 4 \Leftrightarrow f(2) \leq 7.$$

In conclusion, the greatest value that $f$ may attain at 0 is 7.

4) [Bonus] Is the following statement true of false? Justify your answer.

Let $f$ be a function such that $f''(x) > 0$ for all $x$, and $f'(2) = 2$. Then $f(2018) > f(2017)$.

**TRUE**

Since $f''(x) > 0$ for all $x$, then $f'(x)$ is an increasing function on $(-\infty, \infty)$. Since $f'(2) = 2$ and $f'$ is increasing, this implies that $f'(x) > 0$ on $(2, \infty)$; in particular $f'(x) > 0$ on $(2, \infty)$. Thus $f$ is increasing on $(2, \infty)$ and, by definition, $f(2018) > f(2017)$.