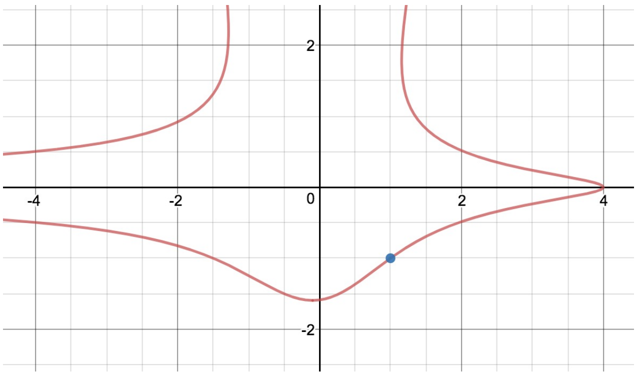
Calculus I - MAC 2311 - Section 001

Quiz 5 - Solutions

02/21/2018

1) [5 points] Consider the curve $\mathcal{C}$ given by the equation

$$x - y^3 = 4 - 2x^2y^2.$$



a) Use implicit differentiation to find y' $\left(\text{i.e. } \frac{dy}{dx}\right)$.

b) Find an equation of the tangent line to the above curve at the point $(1, -1)$.

Solution:

a) We take the derivative of each side of the equation of the curve with respect to x (recall to treat y as a function of x), and apply the rules of differentiation:

$$\frac{d}{dx}\left(x - y^3\right) = \frac{d}{dx}\left(4 - 2x^2y^2\right)$$

$\Downarrow$ sum rule

$$\frac{d}{dx}x - \frac{d}{dx}y^3 = \frac{d}{dx}4 - \frac{d}{dx}(2x^2y^2)$$

$\Downarrow$ product rule+chain rule

$$1 - 3y^2 \cdot \frac{dy}{dx} = 0 - \left[\frac{d}{dx}(2x^2) \cdot y^2 + 2x^2 \cdot \frac{d}{dx}(y^2)\right]$$

$\Downarrow$

$$1 - 3y^2 \cdot \frac{dy}{dx} = -4xy^2 - 4x^2y \cdot \frac{dy}{dx}$$

Now we have an ordinary linear equation where the unknown we want to solve for is $\frac{dy}{dx}$. From the last step we obtain:

$$-3y^2 \cdot \frac{dy}{dx} + 4x^2y \cdot \frac{dy}{dx} = -4xy^2 - 1$$

$\Downarrow$

$$(-3y^2 + 4x^2y) \cdot \frac{dy}{dx} = -4xy^2 - 1$$

which implies

$$\frac{dy}{dx} = \frac{-4xy^2 - 1}{-3y^2 + 4x^2y}.$$

b) If $P(x, y)$ is a point on the lemniscate, i.e. the coordinates x and y of P make the equation of $\mathcal{C}$ true, we have that the slope of the tangent line to the curve $\mathcal{C}$ at $P(x, y)$ is given by:

$$\frac{dy}{dx} = \frac{-4xy^2 - 1}{-3y^2 + 4x^2y}.$$

Hence, for the point $(1, -1)$, by substituting $x = 1$ and $y = -1$ in the previous formula, we get:

$$\frac{dy}{dx} = \frac{-4(1)(-1)^2 - 1}{-3(-1)^2 + 4(1)^2(-1)} = \frac{-4 - 1}{-3 - 4} = \frac{5}{7}.$$

We deduce that an equation of the tangent line to the curve $\mathcal{C}$ at the point $(1, -1)$ is

$$y - (-1) = \frac{5}{7} \cdot (x - 1),$$

i.e.

$$y = \frac{5}{7}x - \frac{12}{7}.$$

2) [5 points] In thermodynamics, **Boyle's law** states that for a fixed amount of an ideal gas kept at a fixed temperature, pressure P and volume V are inversely proportional, i.e.

$$PV = k,$$

where k is a constant. Assume that the quantities P and V depend both on time.

a) Differentiate both sides of Boyle's law to find an equation relating $\frac{dP}{dt}$ and $\frac{dV}{dt}$.

b) A sample of gas is trapped in a cylinder by a piston which is slowly compressed. Suppose that at a certain instant the gas occupies a volume of 60 L (liters) and has a pressure of 50 kPa (kilopascal) and the volume of the gas decreases at a rate of 10 L/min. Assuming the temperature is constant, how quickly is the pressure increasing at this instant?

Solution:

a) Since the volume V and the pressure P depend on time, while k is constant, we can rewrite the Boyle's law in the following way:

$$P(t) \cdot V(t) = k.$$

By differentiating both sides of the previous equation with respect to time we obtain:

$$\frac{d}{dt}\left[P(t) \cdot V(t)\right] = \frac{d}{dt}k$$

$$\Downarrow \text{ product rule}$$

$$\frac{dP}{dt} \cdot V(t) + P(t) \cdot \frac{dV}{dt} = 0$$

b) **Known**: We know that at a certain instant t_0 we have $\left.\frac{dV}{dt}\right|_{t=t_0} = -10$ L/min (the sign "−" is due to the fact that the volume is decreasing), $V(t_0) = 60$ L and $P(t_0) = 50$ kPa.

Unknown: $\frac{dP}{dt}$ at $t = t_0$, i.e. $\left.\frac{dP}{dt}\right|_{t=t_0}$

We have to solve the last equation for $\frac{dP}{dt}$:

$$\frac{dP}{dt} \cdot V(t) + P(t) \cdot \frac{dV}{dt} = 0$$

$$\Downarrow$$

$$\frac{dP}{dt} \cdot V(t) = -P(t) \cdot \frac{dV}{dt}$$

$$\Downarrow$$

$$\frac{dP}{dt} = -\frac{P(t) \cdot \frac{dV}{dt}}{V(t)}$$

At time $t = t_0$ we have:

$$\left.\frac{dP}{dt}\right|_{t=t_0} = -\frac{P(t_0) \cdot \left.\frac{dV}{dt}\right|_{t=t_0}}{V(t_0)} = -\frac{50\text{kPa} \cdot \left(-10\frac{\text{L}}{\text{min}}\right)}{60\text{L}} = \frac{50}{6}\frac{\text{kPa}}{\text{min}} = \frac{25}{3}\,\text{kPa/min}.$$

3) [Bonus] Compute the following derivative:

$$\frac{d}{du}\left[\tan(k^3 u)\right],$$

where k is a constant.

Solution:

Since k is a constant, also k^3 is a constant. Then:

$$\frac{d}{du}\left[\tan(k^3 u)\right] = \sec^2(k^3 u) \cdot \frac{d}{du}(k^3 u) = \sec^2(k^3 u) \cdot k^3.$$