# CURVE SKETCHING (Sec. 4.4).

In the previous classes we saw how we can get information on a function $f$ from the information about its first and second derivatives.

The study of a function consists in the collection of all this information and ends in the curve sketching.

The following steps can be used as guidelines for sketching the graph of a function $f$.

## GUIDELINES FOR SKETCHING A CURVE

1) **DOMAIN**: Find the domain of $f$, i.e. the set of real numbers at which $f$ is defined.
Geometrically, it represents the projection of the graph of $f$ on the $x$-axis.

2) **INTERCEPTS**: The intercepts are the intersections of the graph of $f$ with the $x$-axis and the $y$-axis.

<u>x-intercepts</u>: They are the intersection of the graph $y=f(x)$ with the $x$-axis $(y=0)$

You have to solve the equation $f(x)=0$.

If $x_0$ is a solution of this equation, then $(x_0, f(x_0))$ is a $x$-intercept.

<u>y-intercept</u>: there is at most one $y$-intercept and it occurs when $0$ belongs to the domain of $f$.

It is the intersection of the graph $y=f(x)$ with the $y$-axis $(x=0)$

$\hookrightarrow (0, f(0))$.

3) **SIGN OF THE FUNCTION**: We find the intervals on which the function is positive/negative by solving the inequalities

$$f(x)>0 \quad \text{and} \quad f(x)<0$$

If $f$ is positive (resp. negative) over $(a,b)$ then its graph is above (resp. below) the $x$-axis between $x=a$ and $x=b$.

## 4) SYMMETRY

We check whether the function is odd or even.

If the equality $f(x) = -f(-x)$ holds for all $x$ in the domain of $f$, then the function is <u>odd</u> and the graph is symmetric with respect to the origin $(0,0)$.

If the equality $f(x) = f(-x)$ holds for all $x$ in the domain of $f$ then $f$ is <u>even</u> and the graph is symmetric about the y-axis.

↑ in this case we can study the function on $[0,\infty)$ and then simply reflect the obtained graph about the y-axis.

Note that a function can be neither odd nor even.

## 5) BEHAVIOR AT THE ENDPOINTS OF THE DOMAIN → ASYMPTOTES

We study here the (limit) behaviour of the function at the endpoints of the domain.

For example if the domain of a function is:

$$(-\infty, 1) \cup (1, \infty)$$

we compute $\lim_{x\to-\infty} f(x)$, $\lim_{x\to 1^-} f(x)$, $\lim_{x\to 1^+} f(x)$, $\lim_{x\to\infty} f(x)$.

<u>HORIZONTAL ASYMPTOTE</u>: Recall that $y=L$ is a horizontal asymptote if

either $\lim_{x\to-\infty} f(x) = L$ or $\lim_{x\to\infty} f(x) = L$.

<u>VERTICAL ASYMPTOTE</u>: Recall that $x=a$ is a vertical asymptote if

either $\lim_{x\to a^-} f(x) = \pm\infty$ or $\lim_{x\to a^+} f(x) = \pm\infty$.

## 6) CRITICAL POINTS, INTERVALS OF DECREASE/INCREASE, LOCAL MAXIMUM/MINIMUM VALUES.

- Compute $f'(x)$
- critical points: study where $f'(x)$ is not defined and solve $f'(x) = 0$.
- Find the intervals on which:

  $f'(x) > 0 \Rightarrow f$ increasing.

  $f'(x) < 0 \Rightarrow f$ decreasing.
- First derivative test → local max./min. values.

7) CONCAVITY AND INFLECTION POINTS

- Compute $f''(x)$
- Find the intervals on which

  $f''(x) > 0 \Rightarrow f$ concave up.

  $f''(x) < 0 \Rightarrow f$ concave down.

- Find the inflection points (the points on the graph at which the function is continuous and the concavity changes from up to down or viceversa).

8) SKETCH THE GRAPH BY USING THE COLLECTED INFORMATION.

EXERCISE

Sketch the graph of the function:

$$f(x) = xe^{-x}.$$

1) DOMAIN

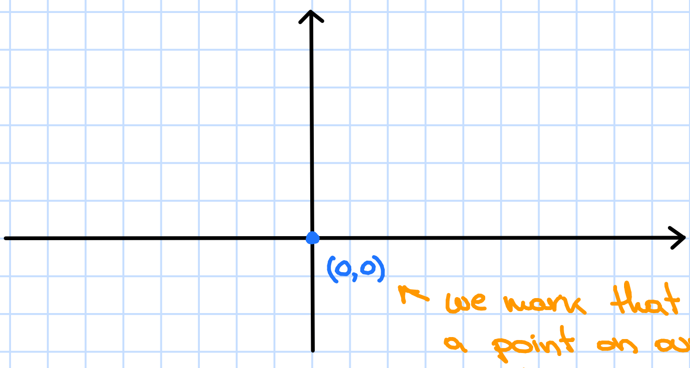
$f$ is defined for all $x \Rightarrow D = \mathbb{R}$

2) INTERCEPTS

x-intercept(s): $f(x) = 0 \Leftrightarrow xe^{-x} = 0 \overset{e^{-x} \neq 0}{\Longleftrightarrow} x = 0.$

$\Rightarrow (0, f(0)) = (0,0)$ is the only x-intercept.

y-intercept: $(0, f(0)) = (0,0)$ is the only y-intercept.

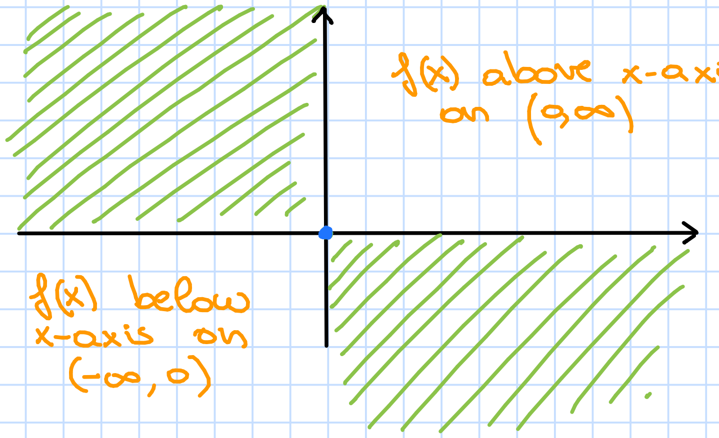



we mark that (0,0) is a point on our "future" graph

3) SIGN OF THE FUNCTION

$f(x) > 0 \Rightarrow xe^{-x} > 0 \overset{e^{-x}>0}{\Longrightarrow} x > 0.$

Hence $f(x) > 0$ on $(0,\infty)$ and $f(x) < 0$ on $(-\infty, 0)$.




4) SYMMETRY

for example:
$f(-1) = e$
$f(1) = \frac{1}{e}$

$f(-x) = -xe^{x}$.

So $f(-x) \neq f(x)$, i.e. $f$ is not even, and $f(-x) \neq -f(x)$, i.e. $f$ is not odd.

5) BEHAVIOUR OF THE FUNCTION AT THE ENDPOINTS OF THE DOMAIN

$D = (-\infty, \infty)$ and $f$ is continuous on $D$.

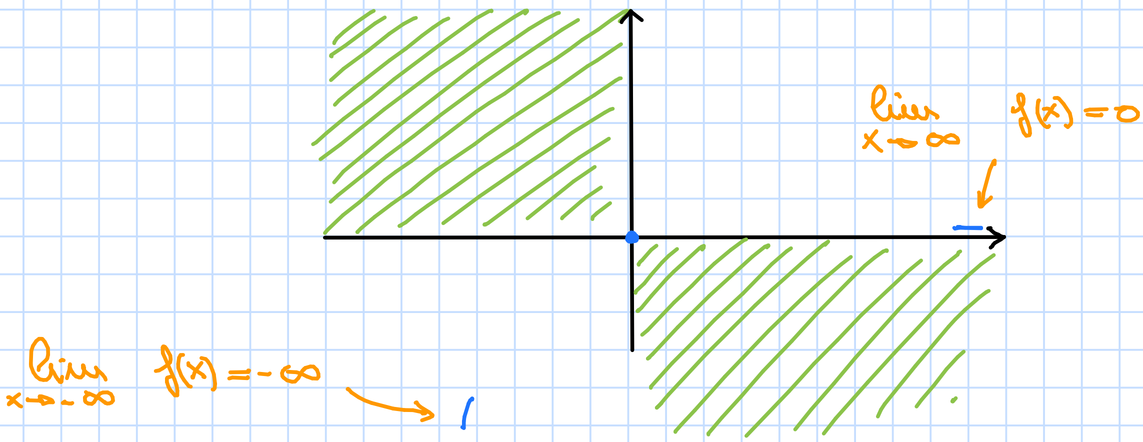
We have:

$$\lim_{x \to -\infty} f(x) = \lim_{x \to -\infty} xe^{-x} = \text{"}-\infty \cdot \infty\text{"} = -\infty$$

$$\lim_{x \to \infty} f(x) = \lim_{x \to \infty} xe^{-x} = \lim_{x \to \infty} \frac{x}{e^x} \underset{\frac{\infty}{\infty} \Rightarrow H.R.}{=} \lim_{x \to \infty} \frac{1}{e^x} = \text{"}\frac{1}{\infty}\text{"} = 0$$

So the line $y=0$ is a horizontal asymptote at $\infty$.

There are no vertical asymptotes.



## 7) CRITICAL POINTS, INTERVAL OF INCREASE / DICREASE, LOCAL MAXIMUM AND MINIMUM VALUES.

$$f'(x) = (x)'e^{-x} + x\cdot(e^{-x})' = e^{-x} - xe^{-x} = e^{-x}(1-x)$$

critical numbers: $f$ is differentiable everywhere, so:

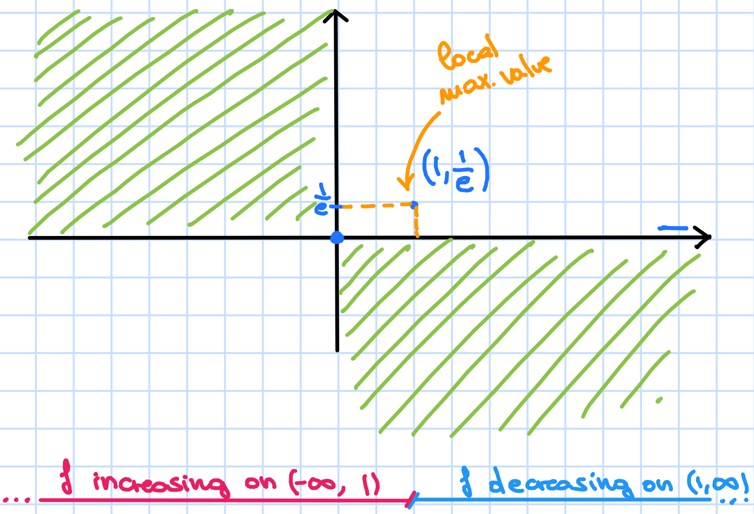
$$f'(x) = 0 \iff e^{-x}(1-x) = 0 \iff (1-x) = 0 \iff x = 1.$$

$e^{-x} \neq 0$

$f'(0) = 1 > 0$ $\qquad$ $f'(2) = e^{-2}\cdot(-1) < 0$

| $x$ | 0 | 1 | 2 |
|---|---|---|---|
| $f'(x)$ | $+$ | | $-$ |
| $f(x)$ | $\nearrow$ | | $\searrow$ |

So $f$ is increasing on $(-\infty, 1)$ and decreasing on $(1, \infty)$.
Moreover $f$ has a local maximum value at $1 \rightarrow f(1) = 1\cdot e^{-1} = \frac{1}{e}$.




## 8) CONCAVITY AND INFLECTION POINTS.

$$f''(x) = [e^{-x}(1-x)]' = (e^{-x})'(1-x) + e^{-x}(1-x)' = -e^{-x}(1-x) + e^{-x}\cdot(-1) =$$
$$= e^{-x}(-1+x-1) = e^{-x}(x-2)$$

$$f''(x) = 0 \iff e^{-x}(x-2) = 0 \Rightarrow x-2 = 0 \iff x = 2$$

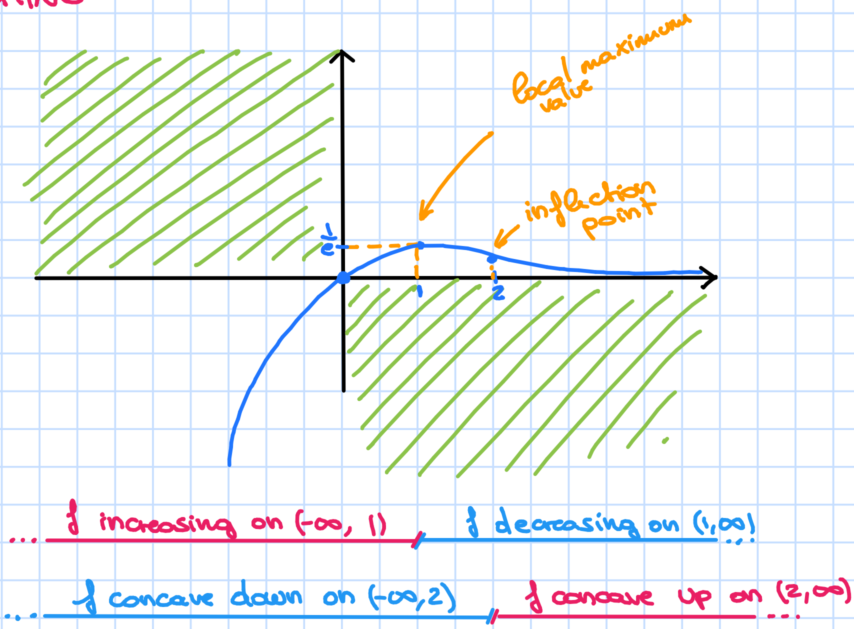
$f''(1) = e^{-1}\cdot(-1) < 0$ $\qquad$ $f''(3) = e^{-3}\cdot 1 > 0$

| $x$ | 1 | 2 | 3 |
|---|---|---|---|
| $f''(x)$ | $-$ | | $+$ |
| $f(x)$ | $\frown$ | | $\smile$ |

Then $f$ is concave down on $(-\infty, 2)$ and concave up on $(2, \infty)$.

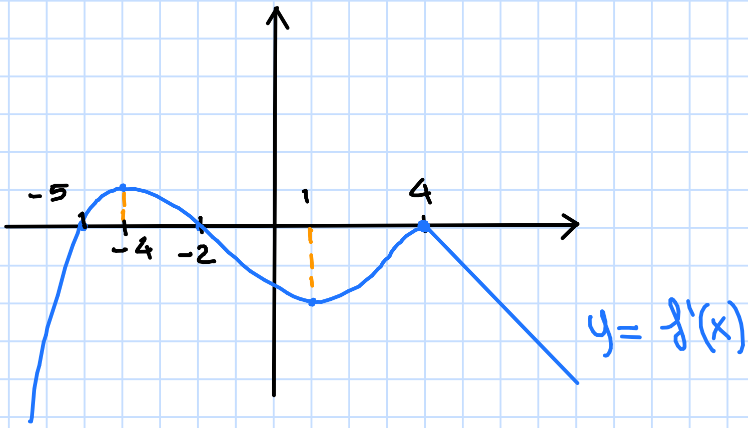
It has an inflection point a $(2, f(2)) = (2, 2e^{-2})$

## 9) CURVE SKETCHING




### EXERCISE

Let $f$ be a function whose derivative has the following graph:




(1) Over which intervals is $f$ increasing / decreasing?

(2) What are the critical numbers of $f$?

(3) At what numbers does $f$ have a local minimum/maximum value?

(4) Over which intervals is $f$ concave down /up?

(5) What are the x-coordinates of the inflection points?

## Solution

First we remark that since the derivative is always defined then $f$ is continuous everywhere.

1) Recall that

- $f'(x) > 0$ on $(a,b) \Rightarrow f$ increasing on $(a,b)$
- $f'(x) < 0$ on $(a,b) \Rightarrow f$ decreasing on $(a,b)$

So we need to study on which intervals $f'$ is positive (i.e. the graph $y = f'(x)$ is above the $x$-axis) and on which intervals $f'$ is negative (i.e. the graph $y = f'(x)$ is below the $x$-axis).

From the graph we have:

$$f'(x) = 0 \iff x = -5 \text{ or } x = -2 \text{ or } x = 4.$$

So we study the sign of $f'(x)$ on the intervals $(-\infty, -5)$, $(-5, -2)$, $(-2, 4)$ and $(4, \infty)$

| | | $-5$ | | $-2$ | | $4$ | |
|---|---|---|---|---|---|---|---|
| $f'(x)$ | $-$ | | $+$ | | $-$ | | $-$ |
| $f(x)$ | $\searrow$ | | $\nearrow$ | | $\searrow$ | | $\searrow$ |

Therefore we have that $f$ is increasing on $(-5, -2)$ and decreasing on $(-\infty, -5) \cup (-2, 4) \cup (4, \infty)$.

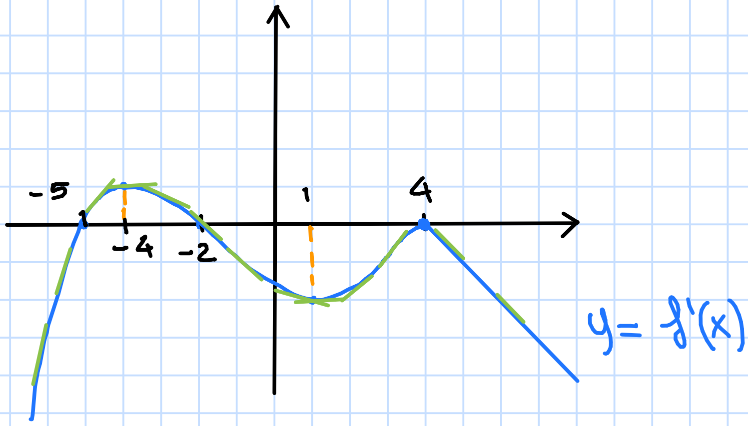
2) From the previous step we get also that the critical numbers of $f$ are $x = -5$, $x = -2$ and $x = 4$ (these are the solutions of the equation $f'(x) = 0$).

3) Again from step 1 we get that $f$ has a local minimum value at $x = -5$ and a local maximum value at $x = -2$.

4) Recall that

- $f''(x) > 0$ on $(a,b) \Rightarrow f$ concave up on $(a,b)$
- $f''(x) < 0$ on $(a,b) \Rightarrow f$ concave down on $(a,b)$

Note that, in the graph of $f'$, the second derivative $f''(x)$ represents the slope of the tangent line to the graph at the point $(x, f'(x))$.




Since $f''(x)$ is zero at $x=-4$ and $x=1$ and it is not defined at $x=4$, we study its sign over the intervals $(-\infty,-4)$, $(-4,1)$, $(1,4)$ and $(4,\infty)$

| | | $-4$ | | $1$ | | $4$ | |
|---|---|---|---|---|---|---|---|
| $f''(x)$ | $+$ | | $-$ | | $+$ | | $-$ |
| $f(x)$ | | | | | | | |

So we have that $f$ is concave up on $(-\infty,-4) \cup (1,4)$ and concave down on $(-4,1) \cup (4,\infty)$

5) From the previous point we have also that $f''(x)$ changes sign at $x=-4$, $x=1$ and $x=4$. Moreover $f$ is continuous everywhere. Then $x=-4$, $x=1$ and $x=4$ are the x-coordinates of the inflection points.