INVERSE TRIGONOMETRIC FUNCTIONS (Sec. 3.5)

Recall that:

If $f: A \to B$ is a one-to-one function, with domain A and range B, then the inverse function f^{-1}:

$$f^{-1}: B \to A$$

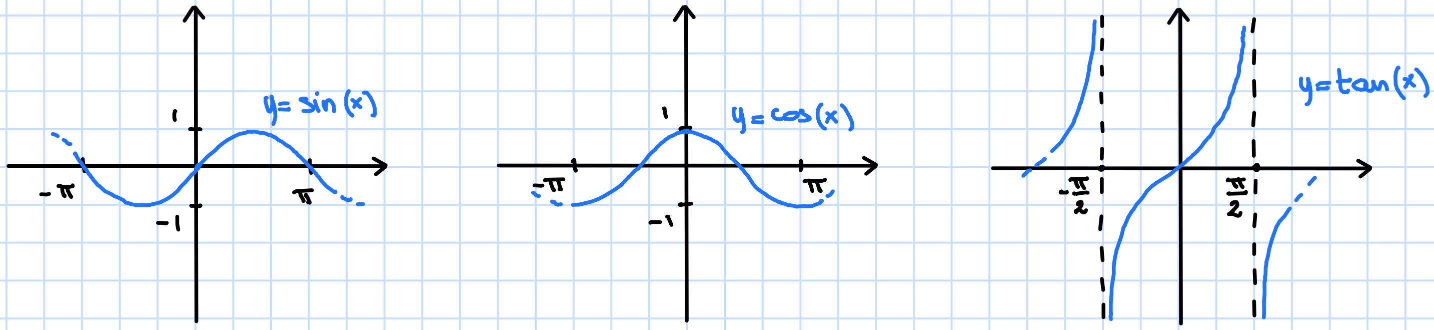
is defined by:

$$\text{for all } x \text{ in } B,\ f^{-1}(x) = y \iff f(y) = x.$$

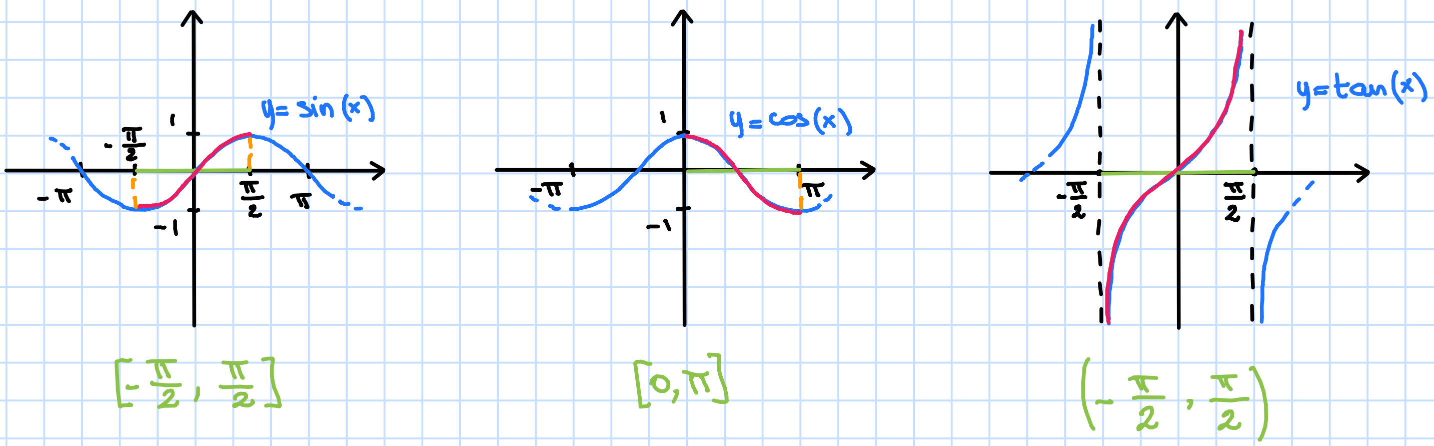
It is easy to note that the trigonometric functions $\sin(x)$, $\cos(x)$, $\tan(x)$ are not one-to-one, since they are periodic functions.

Nevertheless, if we restrict their domain, we can make them one-to-one and then invertible.

Let us have a look to their graphs:



For each of these functions we want to find one of the largest intervals (there are infinitely many choices) on which the function is one-to-one.



We consider then the restrictions of $\sin(x)$, $\cos(x)$ and $\tan(x)$ to these domains:

$\sin : \left[-\frac{\pi}{2}, \frac{\pi}{2}\right] \to [-1, 1]$

$\cos : [0, \pi] \to [-1, 1]$

$\tan : \left(-\frac{\pi}{2}, \frac{\pi}{2}\right) \to \mathbb{R}$

The functions are now one-to-one on these restricted domains

and we define the respective inverse trigonometric functions $\sin^{-1}$ (or arcsin), $\cos^{-1}$ (or arccos) and $\tan^{-1}$ (or arctan) to be:

$\sin^{-1}$ (arcsin) : $[-1, 1] \to \left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$

$$\sin^{-1}(x) = y \iff \sin(y) = x \text{ and } y \in \left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$$

(this condition is essential for y to be unique)

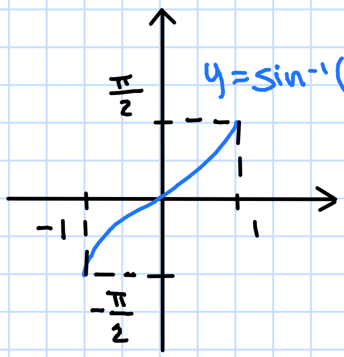
$\cos^{-1}$ (arccos) : $[-1, 1] \to [0, \pi]$

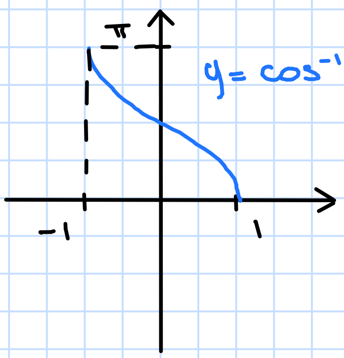
$$\cos^{-1}(x) = y \iff \cos(y) = x \text{ and } y \in [0, \pi]$$

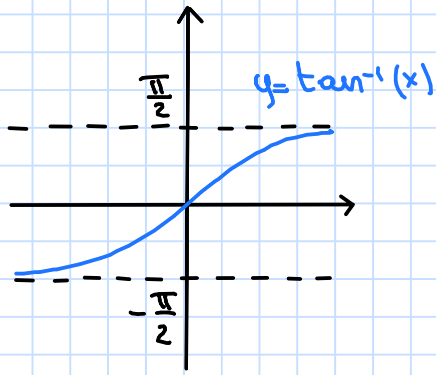
$\tan^{-1}$ (arctan) : $\mathbb{R} \to \left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$

$$\tan^{-1}(x) = y \iff \tan(y) = x \text{ and } y \in \left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$$

The graphs of $\sin^{-1}(x)$, $\cos^{-1}(x)$ and $\tan^{-1}(x)$ are obtained by reflection respectively of the graphs of $\sin(x)$, $\cos(x)$ and $\tan(x)$ (on the restricted domain) about the line $y = x$.







$$\lim_{x \to \infty} \tan^{-1}(x) = \frac{\pi}{2}$$

$$\lim_{x \to -\infty} \tan^{-1}(x) = -\frac{\pi}{2}$$

Examples

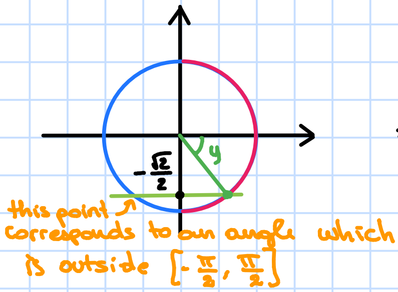
Compute the following values:

- $\sin^{-1}\left(-\frac{\sqrt{2}}{2}\right)$

By definition $\sin^{-1}\left(-\frac{\sqrt{2}}{2}\right) = y$ with $\sin(y) = -\frac{\sqrt{2}}{2}$ and $y \in \left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$

In other words we are looking for the angle y in the interval $\left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$ such that $\sin(y) = -\frac{\sqrt{2}}{2}$.

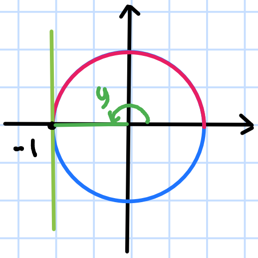
Let us have a look to the unit circle:



$\Rightarrow \sin^{-1}\left(-\frac{\sqrt{2}}{2}\right) = -\frac{\pi}{4}$ (note that $\sin\left(-\frac{\pi}{4}\right) = -\frac{\sqrt{2}}{2}$ and $-\frac{\pi}{4} \in \left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$)

- $\arccos(-1)$

By definition, $\arccos(-1) = y$ with $\cos(y) = -1$ and $y \in [0, \pi]$



$\Rightarrow \arccos(-1) = \pi$

Cancellation equations

The inverse trigonometric functions satisfy the following cancellation equations:

$$\begin{cases} \sin^{-1}(\sin(x)) = x & \text{for all } x \in \left[-\frac{\pi}{2}, \frac{\pi}{2}\right] \\ \sin(\sin^{-1}(x)) = x & \text{for all } x \in [-1, 1] \end{cases}$$

$$\begin{cases} \cos^{-1}(\cos(x)) = x & \text{for all } x \in [0, \pi] \\ \cos(\cos^{-1}(x)) = x & \text{for all } x \in [-1, 1] \end{cases}$$

$$\begin{cases} \tan^{-1}(\tan(x)) = x & \text{for all } x \in \left(-\frac{\pi}{2}, \frac{\pi}{2}\right) \\ \tan(\tan^{-1}(x)) = x & \text{for all } x \in \mathbb{R} \end{cases}$$

Warning : • $\cos^{-1}(\cos(2\pi)) \neq 2\pi$

Indeed, since 2π does not belong to the interval $[0,\pi]$, the cancellation equation does not apply.

We have:

$$\cos^{-1}(\cos(2\pi)) = \cos^{-1}(1) = 0$$

the output has to be in the range of $\cos^{-1}$, i.e. $[0,\pi]$

• $\sin(\sin^{-1}(2))$ is undefined!

Indeed 2 does not belong to the domain of $\sin^{-1}$, i.e. $[-1,1]$.

Exercise : Compute $\tan\left(\arcsin\left(\frac{1}{3}\right)\right)$ without the use of a calculator.

Solution

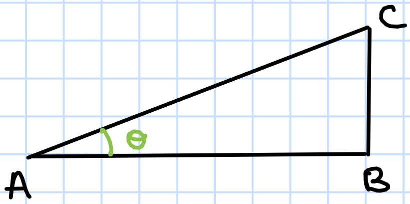
We notice that $\frac{1}{3}$ is not a "remarkable" output for the function $\sin(x)$. So we will adopt a different strategy for solving this exercise.

Let us set $\theta = \arcsin\left(\frac{1}{3}\right)$. By definition, we have:

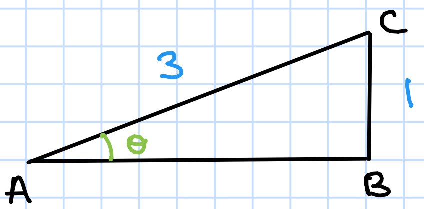
$$\arcsin\left(\frac{1}{3}\right) = \theta \iff \sin(\theta) = \frac{1}{3} \text{ and } \theta \in \left[-\frac{\pi}{2}, \frac{\pi}{2}\right].$$

Now, let us consider a right triangle with an angle equal to θ, where $0 < \theta < \frac{\pi}{2}$.

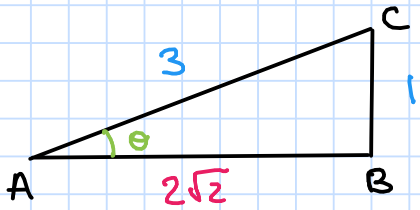
if $\sin(\theta) = \frac{1}{3} \Rightarrow 0 < \theta < \frac{\pi}{2}$



Since $\sin\theta = \frac{1}{3}$ and $\sin\theta = \frac{op}{hy} = \frac{\overline{BC}}{\overline{AC}}$, we can assume $\overline{BC} = 1$ and $\overline{AC} = 3$



By Pythagorean theorem we get $\overline{AB} = \sqrt{3^2 - 1} = \sqrt{8} = 2\sqrt{2}$.



Then $\tan\left(\arcsin\left(\frac{1}{3}\right)\right) = \tan(\theta) = \frac{op}{ad} = \frac{\overline{BC}}{\overline{AB}} = \frac{1}{2\sqrt{2}} = \frac{\sqrt{2}}{4}$

The previous exercise is an example of a more general process of "simplification".

EXAMPLE 1

Simplify the expression $\cos\left(\tan^{-1}(x)\right)$, for all x in $\mathbb{R}$

↓

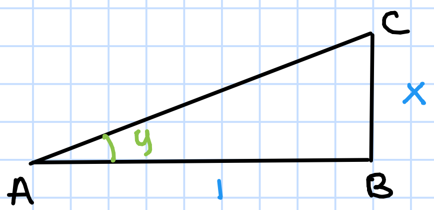
We will proceed analogously to the previous exercise, this time with a generic value x rather than a specific one.

Let us set:

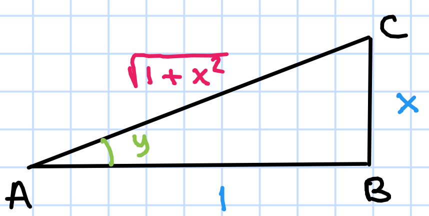
$$y = \tan^{-1}(x) \iff \tan(y) = x \quad \text{and} \quad y \in \left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$$

Now, let us consider a right triangle with an angle equal to y (here, for simplicity we assume $0 \le y < \frac{\pi}{2}$, but the proof can be easily generalized to $-\frac{\pi}{2} < y < \frac{\pi}{2}$).

We have $\tan(y) = x = \frac{x}{1} = \frac{op}{ad}$



By Pythagorean theorem we get $\overline{AC} = \sqrt{1 + x^2}$



Then $\cos(\tan^{-1}(x)) = \cos(y) = \frac{ad}{hy} = \frac{\overline{AB}}{\overline{AC}} = \frac{1}{\sqrt{1+x^2}}$

In conclusion $\cos(\tan^{-1}(x)) = \frac{1}{\sqrt{1+x^2}}$ for all $x \in \mathbb{R}$.

The idea of simplification is that now it is easier to compute the output of the function. For example, for $x = 2$ we have:

$$\cos(\tan^{-1}(2)) = \frac{1}{\sqrt{1+(2)^2}} = \frac{1}{\sqrt{1+4}} = \frac{1}{\sqrt{5}} = \frac{\sqrt{5}}{5}$$

EXAMPLE 2

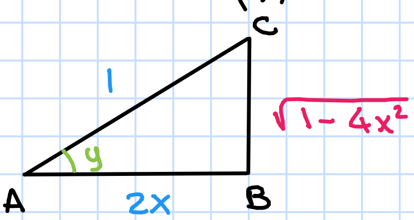
Simplify the expression $\sin(\cos^{-1}(2x))$, for all $x \in [-\frac{1}{2}, \frac{1}{2}]$.

↓

$y = \cos^{-1}(2x) \iff \cos(y) = 2x$ and $y \in [0, \pi]$

For simplicity we assume $0 \leq y < \frac{\pi}{2}$.

We have $\cos(y) = 2x = \frac{2x}{1} = \frac{ad}{hy} = \frac{\overline{AB}}{\overline{AC}}$:



Then $op = \overline{BC} = \sqrt{1-(2x)^2} = \sqrt{1-4x^2}$ and for all $x \in [-\frac{1}{2}, \frac{1}{2}]$:

$$\sin(\cos^{-1}(x)) = \sin(y) = \frac{op}{hy} = \frac{\sqrt{1-4x^2}}{1} = \sqrt{1-4x^2}.$$

Derivatives

We have

① $(\sin^{-1}(x))' = \frac{1}{\sqrt{1-x^2}}$

② $(\cos^{-1}(x))' = -\frac{1}{\sqrt{1-x^2}}$

③ $(\tan^{-1}(x))' = \frac{1}{1+x^2}$

Proof

① Let us set $y = \sin^{-1}(x)$. We want to compute $\frac{dy}{dx}$.

We have

$$y = \sin^{-1}(x) \iff \sin(y) = x \quad \text{and} \quad -\frac{\pi}{2} \leq y \leq \frac{\pi}{2}$$

Now:

$$\frac{d}{dx} \sin(y) = \frac{d}{dx} x$$
$$\Downarrow$$
$$\cos(y) \cdot \frac{dy}{dx} = 1$$
$$\Downarrow$$
$$\frac{dy}{dx} = \frac{1}{\cos(y)} = \frac{1}{\sqrt{1-\sin^2(y)}} = \frac{1}{\sqrt{1-x^2}}$$

- $\sin^2(y) + \cos^2(y) = 1$
- $y \in [-\frac{\pi}{2}, \frac{\pi}{2}] \Rightarrow \cos(y) \geq 0$ (this is the reason why we choose the positive square root)

② $y = \cos^{-1}(x) \iff \cos(y) = x$ and $0 \leq y \leq \pi$

Now:

$$\frac{d}{dx} \cos(y) = \frac{d}{dx} x$$
$$\Downarrow$$
$$-\sin(y) \cdot \frac{dy}{dx} = 1$$
$$\Downarrow$$
$$\frac{dy}{dx} = -\frac{1}{\sin(y)} = -\frac{1}{\sqrt{1-\cos^2(y)}} = -\frac{1}{\sqrt{1-x^2}}$$

- $\sin^2(y) + \cos^2(y) = 1$
- $y \in [0, \pi] \Rightarrow \sin(y) \geq 0$

③ $y = \tan^{-1}(x) \iff \tan(y) = x$ and $-\frac{\pi}{2} < y < \frac{\pi}{2}$

Now:

$$\frac{d}{dx} \tan(y) = \frac{d}{dx} x$$
$$\Downarrow$$
$$\sec^2(y) \cdot \frac{dy}{dx} = 1$$
$$\Downarrow$$
$$\frac{dy}{dx} = \frac{1}{\sec^2(y)} = \frac{1}{\frac{1}{\cos^2(y)}} = \frac{1}{\frac{\cos^2(y) + \sin^2(y)}{\cos^2(y)}} = \frac{1}{1+\tan^2(y)} = \frac{1}{1+x^2}$$

EXERCISE

- Compute the derivative of $\tan^{-1}(x^2+5x+2)$.

$$\left[\tan^{-1}(x^2+5x+2)\right]' = \frac{1}{1+(x^2+5x+2)^2} \cdot (x^2+5x+2)' = \frac{2x+5}{1+(x^2+5x+2)^2}$$

↑ chain rule

- Compute the derivative of $\cos(\tan^{-1}(x))$

$$\left[\cos(\tan^{-1}(x))\right]' = -\sin(\tan^{-1}(x)) \cdot (\tan^{-1}(x))' = -\sin(\tan^{-1}(x)) \cdot \frac{1}{1+x^2}$$

Remark that, since we showed that $\cos(\tan^{-1}(x)) = \frac{1}{\sqrt{1+x^2}}$, then we have also:

$$\left[\cos(\tan^{-1}(x))\right]' = \left[\frac{1}{\sqrt{1+x^2}}\right]' = \left[(1+x^2)^{-\frac{1}{2}}\right]' =$$

$$= -\frac{1}{2}(1+x^2)^{-\frac{3}{2}} \cdot 2x = -\frac{x}{1+x^2} \cdot \frac{1}{\sqrt{1+x^2}}$$

By comparing the two expressions for the derivative of $\cos(\tan^{-1}(x))$ we deduce also that $\sin(\tan^{-1}(x)) = \frac{x}{1+x^2}$.