EXPONENTIAL FUNCTION (Sec. 3.1)

An exponential function is a function of the form

$$f(x) = a^x,$$

where $a > 0$ is a positive real number.

ex: • $f(x) = 2^x, \left(\frac{1}{2}\right)^x, 10^x$ are exponential functions.

• $f(x) = x^2, x^n$ are not exponential functions.

Roughly speaking in an exponential function the variable appears in the exponent. But we have to be careful, since for example the function $f(x) = x^x$ is not an exponential function even if the variable appears (also) at the exponent.

Let us consider now the function $f(x) = 2^x$. What does the value $f(\sqrt{2}) = 2^{\sqrt{2}}$ represent?

For answering to this question let us do some steps back and recall how the operation of exponentiation works.

Recall

Exponentiation is an operation which involves two numbers, a (the base) and n (the exponent):

$$\underset{\nearrow \atop \text{base}}{a}^{n} \leftarrow \text{exponent}$$

When n is a natural integer we define:

$$a^n := \underbrace{a \cdot a \cdots a}_{n \text{ times}} .$$

In particular we have:

$a^0 = 1$ (empty product)

$a^1 = a$

a^2 is the square of a

a^3 is the cube of a.

$$a^{n+1} = a^n \cdot a \quad : \quad a^{n+1} = \underbrace{a \cdots a}_{n+1 \text{ times}} \overset{\substack{\text{associativity} \\ \text{of multiplication} \\ \downarrow}}{=} (\underbrace{a \cdots a}_{n \text{ times}}) \cdot a = a^n \cdot a$$

ex: $2^{11} = \underbrace{2 \cdots 2}_{11 \text{ times}} = 2048$

The operation of exponentiation satisfies the following (important) properties :

① $a^{n+m} = a^n \cdot a^m$ ← this is the fundamental property of exponentiation

$$a^{n+m} = \underbrace{a\cdots a\cdot a\cdots a}_{n+m\text{ times}} \underset{\substack{\uparrow \\ \text{associativity of} \\ \text{multiplication}}}{=} (\underbrace{a\cdots a}_{n\text{ times}})(\underbrace{a\cdots a}_{m\text{ times}}) = a^n\cdot a^m$$

② $(a^n)^m = a^{n\cdot m}$

$$(a^n)^m = a^n\cdots a^n \overset{①}{=} a^{\overbrace{n+\cdots+n}^{m\text{ times}}} = a^{n\cdot m}$$

③ $(ab)^n = a^n b^n$

$$(ab)^n = \underbrace{(ab)(ab)\cdots(ab)}_{n\text{ times}} \underset{\substack{\uparrow \\ \text{commutativity} \\ \text{of multiplication}}}{=} (\underbrace{a\cdots a}_{n\text{ times}})\cdot(\underbrace{b\cdots b}_{n\text{ times}}) = a^n b^n$$

Our goal is now to extend the range of the exponent to negative, rational and real numbers while keeping true the previous properties. This will force us to put some restrictions on the range of the base.

- negative exponent

If n is a natural positive number $(n > 0)$ then

$$a^n\cdot a^{-n} \overset{①}{=} a^{n-n} = a^0 = 1 \overset{a\neq 0}{\Longrightarrow} a^{-n} = \frac{1}{a^n}$$

So we have :

If $a\neq 0$, for all positive natural number $n > 0$ we have $a^{-n} = \frac{1}{a^n}$.

ex : • $2^{-3} = \frac{1}{2^3} = \frac{1}{8}$

• $\left(\frac{1}{2}\right)^{-4} = \frac{1}{\left(\frac{1}{2}\right)^4} = \frac{1}{\frac{1}{16}} = 16$

- rational exponent

If the exponent is a rational number of the form $\frac{1}{n}$, with n an integer we have :

$$\left(a^{\frac{1}{n}}\right)^n = a^{\frac{1}{n}\cdot n} = a^1 = a.$$

So we define $a^{\frac{1}{n}}$ to be the unique real positive solution to the equation $x^n = a$, i.e :

$$a^{\frac{1}{n}} = \sqrt[n]{a} \quad \text{is the principal n-th root of a.}$$

In order for $a^{\frac{1}{n}}$ to be defined for all integers n (both even and odd) we have to assume $a>0$.

We can see that this restriction on the range of the base is fundamental for keeping the properties true, otherwise we can fall in the following paradox:

$$((-1)^2)^{\frac{1}{2}} \overset{②}{=} (-1)^{2\cdot\frac{1}{2}} = (-1)^1 = -1$$

$$(-1)^2 = 1 \longrightarrow ((-1)^2)^{\frac{1}{2}} = (1)^{\frac{1}{2}} = \sqrt{1} = 1$$

$1 = -1$???

Now for a general rational exponent $\frac{p}{q}$ where p and q are integers we have:

$$a^{\frac{p}{q}} = \left(a^{\frac{1}{q}}\right)^p = \left(\sqrt[q]{a}\right)^p = \sqrt[q]{a^p}$$

So we have:

If $a>0$, and $\frac{p}{q}$ is a rational number, then $a^{\frac{p}{q}} = \sqrt[q]{a^p}$

<u>ex</u>:
- $3^{\frac{3}{2}} = \sqrt{3^3} = 3\sqrt{3}$.
- $\left(\frac{1}{8}\right)^{-\frac{2}{3}} = 8^{\frac{2}{3}} = \sqrt[3]{8^2} = \left(\sqrt[3]{8}\right)^2 = 2^2 = 4$

- a real number as an exponent

Coming back to our initial question: how do we define the value of $2^{\sqrt{2}}$?

The idea is that for every real number x we can find a sequence of rational numbers whose limit is x.

For example if $x = \sqrt{2}$, we know that

$$\sqrt{2} = 1.414213562...$$

So we have :

this is a sequence of rational number approximating $\sqrt{2}$ from the left

① $1.4 < \sqrt{2} < 1.5$

② $1.41 < \sqrt{2} < 1.42$

③ $1.414 < \sqrt{2} < 1.415$

$\vdots \qquad \vdots$

this is a sequence of rational numbers approximating $\sqrt{2}$ from the right.

Now, for each rational number r bounding $\sqrt{2}$ from the left or from the right we are able to compute 2^r:

$$\sqrt[10]{2^{14}} = 2^{\frac{14}{10}} =$$

① $$2.639015\ldots = \boxed{2^{1.4}} < 2^{\sqrt{2}} < 2^{1.5} = 2.828427\ldots$$

② $$2.6573716\ldots = 2^{1.41} < 2^{\sqrt{2}} < 2^{1.42} = 2.6758551\ldots$$

③ $$2.6647496\ldots = 2^{1.414} < 2^{\sqrt{2}} < 2^{1.415} = 2.66659735\ldots$$

$$\vdots \qquad \vdots$$

We note that the numbers on the left and the numbers on the right converge to a same number.

So we define $2^{\sqrt{2}}$ to be the limit of 2^r, where r is a rational number approaching $\sqrt{2}$:

$$2^{\sqrt{2}} = \lim_{\substack{r \to \sqrt{2} \\ r \text{ rational} \\ \text{number}}} 2^r .$$

More in general, if $x \in \mathbb{R} \setminus \mathbb{Q}$, i.e. x is a real but non rational number, and $a > 0$ then we define:

$$a^x = \lim_{\substack{r \to x \\ r \text{ rational} \\ \text{number}}} a^r$$

In this way we have extended the range of the exponent to all the real numbers and we call the function:

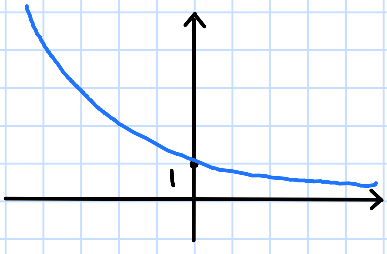
$$f : \mathbb{R} \longrightarrow \mathbb{R}$$
$$x \longmapsto a^x$$

i.e.

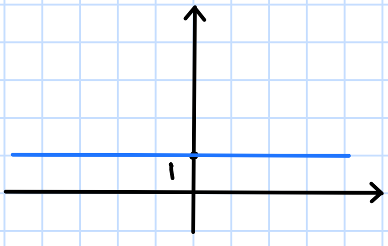
$$f(x) = a^x$$

the exponential function with base a.

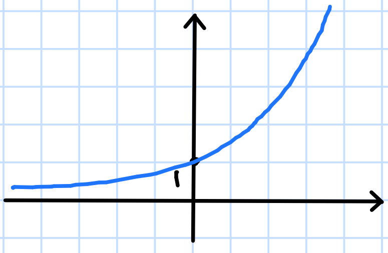
The graph of an exponential function has a different shape depending on the value of the base a:



$0 < a < 1$
$y = a^x$



$a = 1$
$y = 1^x = 1$



$a > 1$
$y = a^x$

Theorem: If $a>0$ and $a\neq 1$ then $f(x)=a^x$ is a continuous function with domain $\mathbb{R}$ and range $(0,\infty)$.

Moreover $f(x)$ satisfies the following laws.

LAWS OF EXPONENTIAL

If $a,b>0$ and $x,y\in\mathbb{R}$ then:

- $a^0=1$
- $a^1=1$
- $a^{x+y}=a^xa^y$
- $a^{x-y}=\frac{a^x}{a^y}$
- $(a^x)^y=a^{xy}$
- $(ab)^x=a^xb^x$

Note that the previous theorem (the continuity and the laws) follows by the way in which we extended the range of the exponent.

Indeed at each step ($\mathbb{N}\to\mathbb{Z}$, $\mathbb{Z}\to\mathbb{Q}$, $\mathbb{Q}\to\mathbb{R}$) the "extension" was defined by using the properties of exponentiation, so that they stay true.
The definition of the exponential function at real not rational numbers as a limit guarantees the continuity.

Which is the "most convenient" base?

We saw that for every real number $a>0$ we can define an exponential function with base a. The question is now: is there a "most convenient" choice on the base among all the positive real numbers?

The answer is yes, but the reasons will be clear later.

This "special base" is the number e, which is also called Euler's number (from the Swiss mathematician Euler) or Napier's constant.

The number e is defined as the limit of a function:

$$e:=\lim_{x\to 0}(1+x)^{\frac{1}{x}}\cong 2.71828\ldots$$

It was discovered by Bernoulli in 1683, but the symbol was introduced by Euler. It is the most important constant in mathematics after π and it shares with π the fact of being an irrational and transcendental number.

There exists a beautiful equation relating all the most important mathematical constants: 0, 1, i, e, π.

algebra geometry

EULER'S IDENTITY: $e^{i\pi} + 1 = 0$ MATH

analysis arithmetic

If we consider each of the constants as a representative of a different branch of mathematics (algebra, geometry, analysis and arithmetic), the identity seems linking them all together in a marvelously simple manner.

Natural exponential function

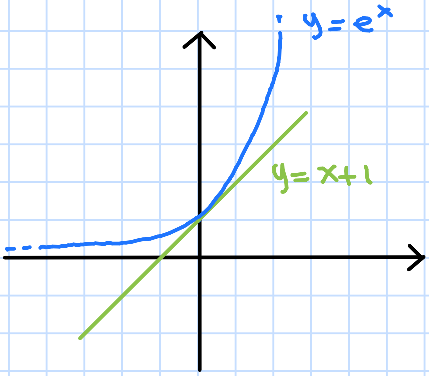
The exponential function with base e is called natural exponential function:

$$f(x) = e^x .$$

Since $2 < e < 3$, we have that $2^x < e^x < 3^x$.

Properties of $f(x) = e^x$

- domain: $\mathbb{R}$
- range: $(0, \infty)$, $e^x > 0$ for all $x \in \mathbb{R}$
- continuous everywhere
- $\lim_{x\to\infty} e^x = \infty$
- $\lim_{x\to-\infty} e^x = 0 \Rightarrow y = 0$ horizontal asymptote



We will also see that e^x is the only one among all the exponential functions having tangent line of slope 1 at $x=0$.