DERIVATIVES AND RATE OF CHANGE (Sec 2.1)

We enter now the first branch of calculus, which is differential calculus.

The fundamental notion of differential calculus is the derivative, that measures the sensitivity to change of a function value with respect to a change in its argument.

We will see that in order to define the derivative of a function we need the concept of limit.

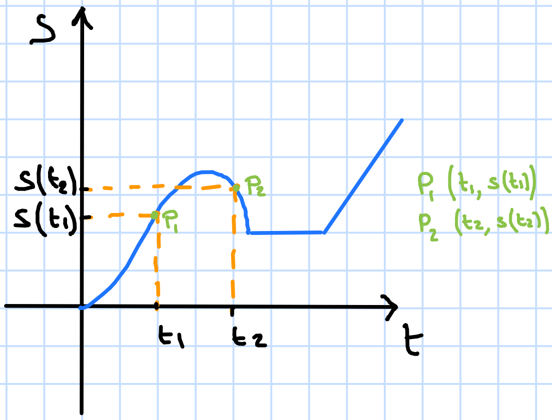
Let us consider three apparently different problems:

- velocity;
- tangent line;
- rate of change.

We will see that the derivative generalizes each one of these concepts.

1) VELOCITY PROBLEM

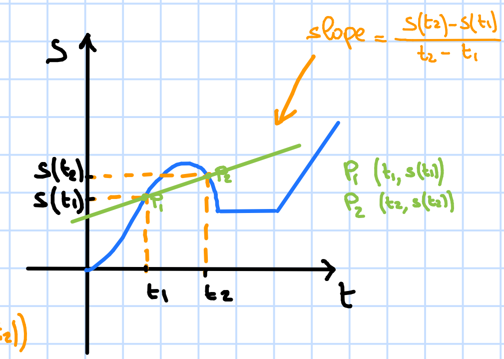
Let $s(t)$ be a position function (= function of the position of an object with respect to time) whose graph is the following:



Recall that the average velocity between t_1 and t_2 is given by:

$$\frac{\text{displacement}}{\text{time}} = \frac{s(t_2)-s(t_1)}{t_2-t_1}$$

note that this is the slope of the secant line through the points $(t_1, s(t_1))$ and $(t_2, s(t_2))$



Imagine that now we want to compute the velocity exactly at t_1, i.e. the instantaneous velocity at $t=t_1$.

Behind the adjective "instantaneous" there is the notion of limit.

Indeed if we can not apply the previous formula since we would get:

$$\frac{s(t_1)-s(t_1)}{t_1-t_1} = \frac{0}{0}$$

But we can see the instantaneous velocity as the limit of the average velocity when t_2 approaches t_1.

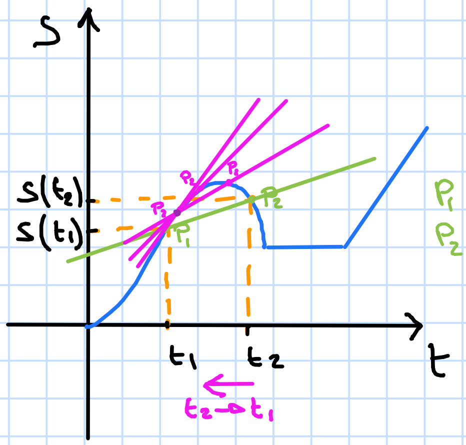
Hence, if $v(t)$ represents the instantaneous velocity at each time t, we have:

$$v(t_1) = \lim_{t_2 \to t_1} \frac{s(t_2)-s(t_1)}{t_2-t_1}$$

or equivalently we can write:

$$v(t_1) = \lim_{t \to t_1} \frac{s(t)-s(t_1)}{t-t_1}$$

Geometrically, when $t_2 \to t_1$ the point P_2 on the graph is approaching P_1 and the secant line through P_1 and P_2 approaches more and more the tangent line to the graph at P_1



average velocity between t_1 and t_2

$$\frac{s(t_2)-s(t_1)}{t_2-t_1}$$

slope of the secant line through $P_1(t_1, s(t_1))$ and $P_2(t_2, s(t_2))$

$\xrightarrow[t_2 \to t_1]{\text{limit}}$

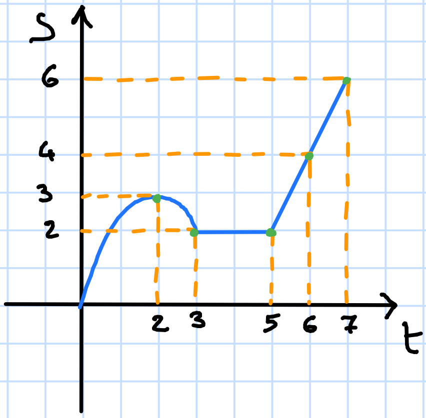
instantaneous velocity at t_1

$$\lim_{t_2 \to t_1} \frac{s(t_2)-s(t_1)}{t_2-t_1}$$

slope of the tangent line to the graph at $P_1(t_1, s(t_1))$

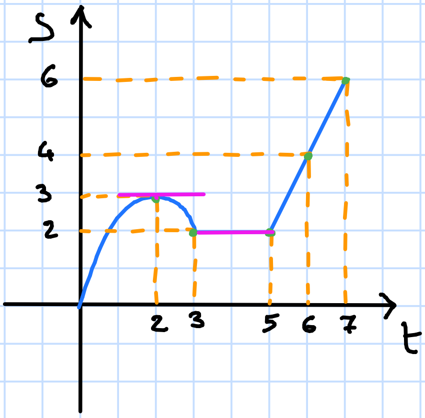
This implies that, if the graph of a position function is given, then we can see the instantaneous velocity at each time t as the slope of the tangent line to the graph at the point $(t, s(t))$.

ex: Let us consider the following graph of a position function s(t), where position is measured in meters and time in seconds.



a) When is the (instantaneous) velocity zero?

We have to find the points of the graph at which the tangent line is horizontal:



This is true for the point of coordinates (2,3) and for all the points with t-coordinate between 3 and 5.
(note that the tangent line to a line is the line itself)

here it is important that the interval is open

Thus, the velocity is 0 when $\underbrace{t=2 \text{ or } 3<t<5}$.

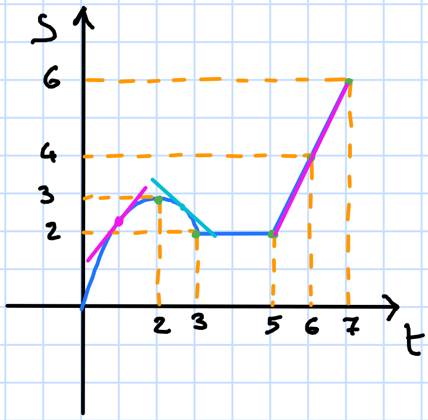
these are the t-coordinates of the previous points!

b) When is the (instantaneous) velocity positive (negative)?

We have to find the points of the graph at which the tangent line has positive (negative slope

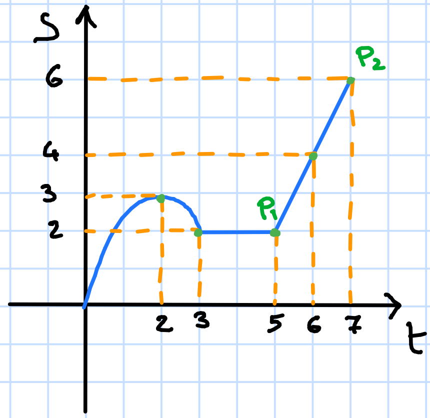
Positive: $0<t<2$ or $5<t<7$

Negative: $2<t<3$



c) What is the velocity at $t = 6$ s?

The instantaneous velocity at $t = 6s$ is given by the slope of the tangent line to the graph at the point $(6,4)$.



Since at that point the graph of the position function is a line, then the tangent line is the line itself, and all we have to do is computing its slope.

For that we need the coordinates of two points of the line:

$$P_1(5,2),\ P_2(7,6) \Rightarrow \text{slope} = \frac{6-2}{7-5} = \frac{4}{2} = 2$$

Then the instantaneous velocity at $t=2$ s is 2 m/s.

Remark: Since between 5s and 7s the graph of the position function is a line, then for all times $5 < t < 7$ the instantaneous velocity at t corresponds to the average velocity between $t=5$s and $t=7$s.

This means that the velocity is constant between $t=5$s and $t=7$s.

ex: If $s(t) = 4t+2$ is a position function, compute the instantaneous velocity at $t=1$.

Solution

We could solve this problem geometrically, since the graph of $s(t)$ is a line.

But we will follow here an algebraic way.

Recall the previous formula:

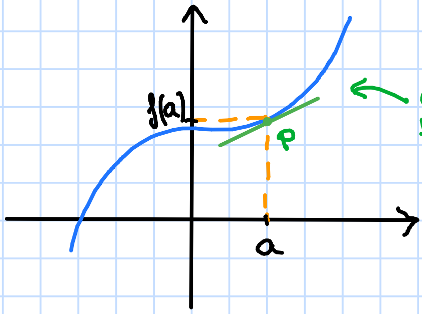
$$v(t_1) = \lim_{t \to t_1} \frac{s(t)-s(t_1)}{t-t_1}.$$

In our case we want to compute the instantaneous velocity at $t=1$. Then:

$$v(1) = \lim_{t \to 1} \frac{s(t)-s(1)}{t-1} = \lim_{t \to 1} \frac{4t+2-(4+2)}{t-1} = \lim_{t \to 1} \frac{4t-4}{t-1} = \lim_{t \to 1} \frac{4\cancel{(t-1)}}{\cancel{t-1}} = 4$$

2) TANGENT PROBLEM

Problem: Given a function $f(x)$, find an equation of the tangent line to the graph $y=f(x)$ at the point $(a, f(a))$



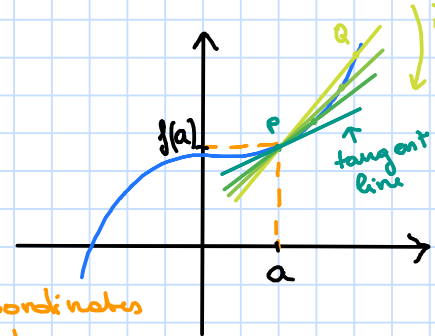
Recall that for finding an equation of the tangent line we need a point and the slope of the line.

In this case the point is given: $P(a, f(a))$.

So the previous problem is equivalent to the following one:

Problem: Given a function $f(x)$, find the slope of the tangent line to the graph $y=f(x)$ at the point $(a, f(a))$.

We can see the tangent line at P as a limit of the secant line through P and Q when Q approaches P on the graph.



If $P(a, f(a))$ and $Q(x, f(x))$ then we have:

generic coordinates for a point on the graph $y=f(x)$

slope of the tangent line at P: $m = \lim_{Q \to P} \frac{y_Q - y_P}{x_Q - x_P} = \lim_{x \to a} \frac{f(x)-f(a)}{x-a}$

$Q(x, f(x))$
$P(a, f(a))$

Hence an equation of the tangent line to the graph $y=f(x)$ at $P(a, f(a))$ is:

$$y - f(a) = m(x-a)$$

To recap:

The tangent line to the curve $y = f(x)$ at the point $P(a, f(a))$ is the line through P with slope:

$$m = \lim_{x \to a} \frac{f(x) - f(a)}{x - a}$$

ex: Find an equation of the tangent line to the graph of $f(x) = x^2$ at the point $(1,1)$.

Solution

The slope of the tangent line is given by the above formula where $a=1$:

$$m = \lim_{x \to 1} \frac{f(x) - f(1)}{x - 1} = \lim_{x \to 1} \frac{x^2 - 1}{x - 1} = \lim_{x \to 1} \frac{(x+1)\cancel{(x-1)}}{\cancel{x-1}} = 2$$

Hence an equation is given by:

$$y - f(1) = m(x-1) \iff y - 1 = 2(x-1) \iff y = 2x - 1.$$

3) RATE OF CHANGE

For a function which is not a position function the equivalent of "average velocity" is the average rate of change.

Let $f(x)$ be a function. If

the values of y depends on the values of x.

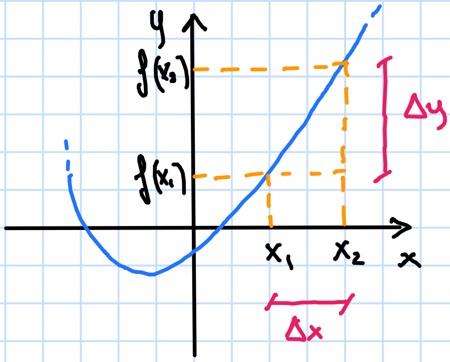
$\Delta x =$ increment of x (independent variable)

$\Delta y =$ increment of y (dependent variable)

Then the average rate of change of y with respect to x is:

dependent variable

independent variable



average rate of change: $\dfrac{\Delta y}{\Delta x} = \dfrac{f(x_2) - f(x_1)}{x_2 - x_1}$

Again, for passing to the instantaneous rate of change we have to take the limit:

instantaneous rate of change: $\displaystyle\lim_{\Delta x \to 0} \frac{\Delta y}{\Delta x} = \lim_{x_2 \to x_1} \frac{f(x_2) - f(x_1)}{x_2 - x_1}$

In particular we can see the instantaneous velocity as the instantaneous rate of change of a position function

---

The concept of "derivative" is nothing else than a generalization of these three concepts:

- instantaneous velocity
- slope of the tangent line
- instantaneous rate of change.

Def: The derivative of a function f at a number a, denoted $f'(a)$, is:

$$f'(a) = \lim_{x \to a} \frac{f(x) - f(a)}{x - a} \overset{h=x-a}{=} \lim_{h \to 0} \frac{f(a+h) - f(a)}{h}$$

if this limit exists.

Note that for passing from the left to the right hand side we did a substitution

If $h = x - a$ then:
1) $x = a + h$
2) when x approaches a then h approaches 0 (since $\lim_{x \to a} h = \lim_{x \to a} x - a = 0$)

IMPORTANT REMARKS

1) Geometrically the derivative $f'(a)$ represents the slope of the tangent line to the graph $y = f(x)$ at the point $(a, f(a))$, and an equation for this tangent line is given by:

$$y - f(a) = f'(a)\,(x - a)$$

2) The derivative $f'(a)$ is the instantaneous rate of change of $y = f(x)$ with respect to x when $x = a$.

3) If f is a position function, then $f'(a)$ is the instantaneous velocity at $t = a$.