LIMITS INVOLVING INFINITY (Sec. 1.6)

In class 2 we built a table of values for the function $\frac{1}{x^2}$ when x approaches 0:

x	$\frac{1}{x^2}$
± 1	1
± 0.5	4
± 0.2	25
± 0.1	100
± 0.05	400
± 0.01	10,000
± 0.001	1,000,000
$\downarrow$ 0	

and we remarked that, while x approaches 0, then $\frac{1}{x^2}$ becomes arbitrarily large.

We denote this situation by:

$$\lim_{x \to 0} \frac{1}{x^2} = \infty \rightarrow$$

this means that the values of $\frac{1}{x^2}$ can be made arbitrarily large by taking x sufficiently close to 0 (on either side of 0) but not equal to 0.

Curiosity: The symbol "∞" was introduced by John Wallis in 1655 in his book "De sectionibus conicis".

There are several hypothesis about the origin of this symbol: the most accredited is that ∞ is a variant of a Roman numeral 1,000 (originally C I Ɔ) which was sometimes used to mean "many".

Analogously the writing:

$$\lim_{x \to a} f(x) = -\infty$$

denotes that the values of $f(x)$ are as large negative as we like for all values of x that are sufficiently close to a, but not equal to a.

Let us consider now the following limit: $\lim_{x \to 0} \frac{1}{x}$.

We note that the output of the function $\frac{1}{x}$ is "very different" when x is close to 0 from the left and from the right

FROM THE LEFT

x	$\frac{1}{x}$
-1	-1
-0.5	-2
-0.2	-5
-0.1	-10
-0.05	-20
-0.01	-100
-0.001	-1000
$\downarrow$	$\downarrow$
0^-	$-\infty$

FROM THE RIGHT

x	$\frac{1}{x}$
1	1
0.5	2
0.2	5
0.1	10
0.05	20
0.01	100
0.001	1000
$\downarrow$	$\downarrow$
0^+	∞

Hence we have:

$$\lim_{x\to 0^-} \frac{1}{x} = \text{"}\frac{1}{0^-}\text{"} = -\infty$$

a positive quantity divided by a very small negative quantity gives a very large negative number

$$\lim_{x\to 0^+} \frac{1}{x} = \text{"}\frac{1}{0^+}\text{"} = \infty$$

$\Rightarrow \lim_{x\to 0} \frac{1}{x}$ does not exist

a positive quantity divided by a very small positive quantity gives a very large positive number

Recap: Each time that the "plug in" returns $\frac{1}{0}$ (or more in general $\frac{L}{0}$, with $L \neq 0$) we have to compute separately the left-hand and the right hand limit.

The value of the one-sided limits will be ∞ or $-\infty$ depending on the sign of the denominator.

We denote by 0^+ a very small (= close to 0) positive quantity and by 0^- a very small negative quantity and we have:

$$\frac{1}{0^+} = \infty \quad \text{and} \quad \frac{1}{0^-} = -\infty$$

More in general:

$\frac{L}{0^+} = \infty$ if $L > 0$

$\frac{L}{0^-} = -\infty$ if $L < 0$

$\frac{L}{0^+} = -\infty$ if $L < 0$

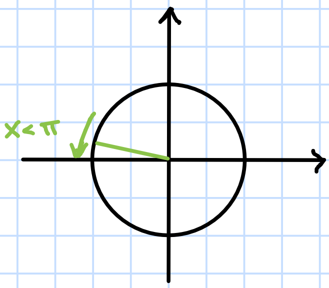
$\frac{L}{0^-} = +\infty$ if $L > 0$

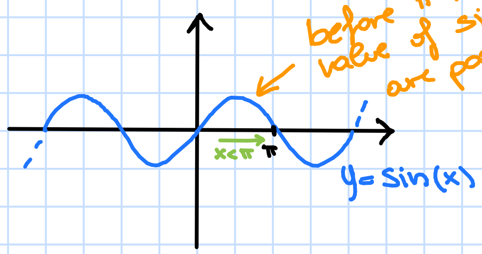
Example: $\lim\limits_{x\to\pi^-} \dfrac{\cos(x)}{\sin(x)} \underset{\substack{\uparrow\\ \text{plug in}}}{=} \dfrac{\cos(\pi)}{\sin(\pi)} = \text{"}\dfrac{-1}{0}\text{"}$

Hence the value of the limit will be ∞ or $-\infty$ depending on the sign of $\sin(x)$ when x approaches π from the left.

Let us consider the unit circle (or the graph of $\sin(x)$)

$\sin(x)$ is positive for angles close to π but less than π





When $x\to\pi^-$ $(x<\pi)$ then $\sin(x)\to 0^+$ ($\sin x$ is close to 0 and positive).

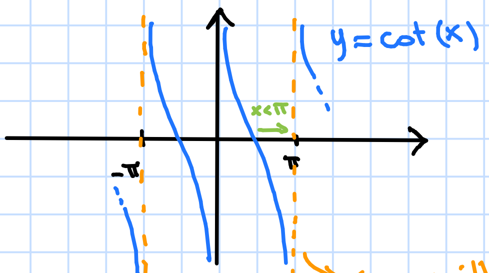
Hence

$\lim\limits_{x\to\pi^-} \dfrac{\cos(x)}{\sin(x)} = \text{"}\dfrac{-1}{0^+}\text{"} = -\infty$

We can also plug in a value $<\pi$ for checkin the sign:

ex: $\dfrac{\cos(3.1)}{\sin(3.1)} \sim -24 < 0$

We could have achieved the same conclusion by remarking that $\dfrac{\cos(x)}{\sin(x)} = \cot(x)$ and by looking at the graph of the $\cot(x)$:



we will see that these orange vertical lines $(x=-\pi,\ x=\pi)$ are called vertical asymptotes.

ex: $\lim\limits_{x\to 2} \dfrac{x-3}{(x-2)^2} \underset{\substack{\uparrow\\ \text{plug in}}}{=} \text{"}\dfrac{-1}{0}\text{"}$

$\lim\limits_{x\to 2^+} \dfrac{x-3}{(x-2)^2} \underset{\uparrow}{=} \text{"}\dfrac{-1}{0^+}\text{"} = -\infty$

$(x-2)^2 > 0$ (a square is always positive)

$$\lim_{x\to 2^-} \frac{x-3}{(x-2)^2} = \text{“}\frac{-1}{0^+}\text{”} = -\infty$$

$(x-2)^2>0$

Thus $\lim_{x\to 2} \frac{x-3}{(x-2)^2} = -\infty$.

So far we have always computed a limit when x approaches a number.

But we can also compute limits for x approaching ∞ or $-\infty$, i.e. when x is arbitrarily large (positive or negative).

Let us start from a very easy case: $\lim_{x\to\infty} \frac{1}{x}$

We have: $\text{“}\frac{1}{\infty}\text{”} = 0$.

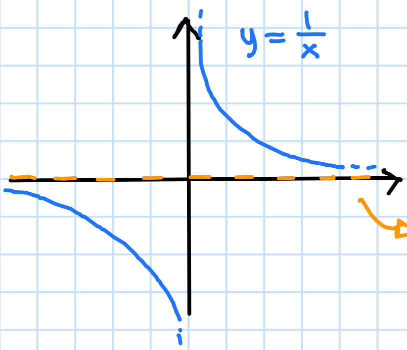
To convince yourself you can construct a table of values where x is very large (1000; 1000,000; etc...) or you can just imagine that you want to share fairly a PIZZA with many many people! How much pizza will everyone get?

when the number of people is very very big the quantity pizza available for everyone is pratically 0!

Conclusion: never share your pizza!

So we have $\lim_{x\to\infty} \frac{1}{x} = 0$ and we have also $\lim_{x\to-\infty} \frac{1}{x} = 0$

This is clear from the graph of $\frac{1}{x}$



we will see that the horizontal line $y=0$ is in this case called a horizontal asymptote

Note that when we compute the limit of a function at ∞ or $-\infty$ we can still plug in, by applying the following rules:

SUM

$L+\infty=\infty$

$L-\infty=-\infty$

$\infty+\infty=\infty$

$-\infty-\infty=-\infty$

$\infty-\infty=$ indeterminate form!
↓
we can not say anything!

PRODUCT

$L>0,\ L\cdot\infty=\infty$

$L<0,\ L\cdot\infty=-\infty$

$L>0,\ L\cdot(-\infty)=-\infty$

$L<0,\ L\cdot(-\infty)=-\infty$

$\infty\cdot\infty=\infty$

$\infty\cdot(-\infty)=-\infty$

$(-\infty)\cdot(-\infty)=\infty$

$+\cdot+=+$
$+\cdot-=-$
$-\cdot-=+$

$0\cdot\infty,\ 0\cdot(-\infty)=$ indeterminate form!
↓
we can not say anything!

QUOTIENT

$\dfrac{L}{\pm\infty}=0$

$L>0,\ \dfrac{\infty}{L}=\infty$

$L<0,\ \dfrac{\infty}{L}=-\infty$

$L>0,\ \dfrac{-\infty}{L}=-\infty$

$L<0,\ \dfrac{-\infty}{L}=+\infty$

$\dfrac{\infty}{\infty},\ \dfrac{-\infty}{\infty},\ \dfrac{\infty}{-\infty},\ \dfrac{-\infty}{-\infty}=$ indeterminate form!
↓
we can not say anything!

POWER / ROOT

n integer, $\infty^n=\infty$

n even, $(-\infty)^n=\infty$

n odd, $(-\infty)^n=-\infty$

n integer, $\sqrt[n]{\infty}=\infty$

n odd, $\sqrt[n]{-\infty}=-\infty$

Example: $\lim_{x\to\infty} x^2-x =$ "$\infty^2-\infty$" $=$ "$\infty-\infty$" : INDETERMINATE FORM

We can escape to the indeterminate form in the following way

$$\lim_{x\to\infty} x^2-x=\lim_{x\to\infty} x(x-1)=\text{"}\infty(\infty-1)\text{"}=\text{"}\infty\cdot\infty\text{"}=\infty$$

Limit at ∞ or $-\infty$ of a rational function

$\lim_{x\to\pm\infty} \frac{P(x)}{Q(x)}$, where $P(x)$ and $Q(x)$ are polynomials.

For computing the limit of a rational function at $+\infty$ or $-\infty$ the technique is standard:

you have to factor the numerator and the denominator respectively by their higher power of x.

This will be more clear on some examples:

if $P(x) = a_n x^n + \cdots + a_0$ with $a_n \neq 0$ then $\deg P = n$ (ex: $\deg(x^7+2x+1)=7$)

ex 1: $\deg(P) = \deg(Q)$

$$\lim_{x\to\infty} \frac{3x^2-x-2}{5x^2+4x+1} = \lim_{x\to\infty} \frac{x^2\left(\frac{3x^2}{x^2}-\frac{x}{x^2}-\frac{2}{x^2}\right)}{x^2\left(\frac{5x^2}{x^2}+\frac{4x}{x^2}+\frac{1}{x^2}\right)} =$$

$$= \lim_{x\to\infty} \frac{\cancel{x^2}\left(3-\frac{1}{x}-\frac{2}{x^2}\right)}{\cancel{x^2}\left(5+\frac{4}{x}+\frac{1}{x^2}\right)} = \lim_{x\to\infty} \frac{3-\frac{1}{x}-\frac{2}{x^2}}{5+\frac{4}{x}+\frac{1}{x^2}} =$$

$$= \text{“}\frac{3-\frac{1}{\infty}-\frac{2}{\infty}}{5+\frac{4}{\infty}+\frac{1}{\infty}}\text{”} = \frac{3-0-0}{5+0+0} = \frac{3}{5}$$

recall $\frac{L}{\infty}=0$

note that this is the ratio of the leading coefficients of P and Q.

ex 2: $\deg(P) < \deg(Q)$

$$\lim_{x\to\infty} \frac{x^2+1}{4x^3+5x-4} = \lim_{x\to\infty} \frac{\cancel{x^2}\left(1+\frac{1}{x^2}\right)}{x^{\cancel{3}}\left(4+\frac{5}{x^2}-\frac{4}{x^3}\right)} = \lim_{x\to\infty} \frac{1+\frac{1}{x^2}}{x\left(4+\frac{5}{x^2}-\frac{4}{x^3}\right)} =$$

$$= \text{“}\frac{1+\frac{1}{\infty}}{\infty\left(4+\frac{5}{\infty}-\frac{4}{\infty}\right)}\text{”} = \text{“}\frac{1+0}{\infty\cdot(4+0-0)}\text{”} = \text{“}\frac{1}{\infty\cdot 4}\text{”} = \frac{1}{\infty} = 0$$

ex 2: $\deg(P) > \deg(Q)$

$$\lim_{x\to-\infty} \frac{x^3-1}{-x^2+3} = \lim_{x\to-\infty} \frac{x^{\cancel{3}}\left(1-\frac{1}{x^3}\right)}{\cancel{x^2}\left(-1+\frac{3}{x^2}\right)} = \lim_{x\to-\infty} \frac{x\left(1-\frac{1}{x^3}\right)}{-1+\frac{3}{x^2}} =$$

$$= \text{“}\frac{-\infty\left(1-\frac{1}{-\infty}\right)}{-1+\frac{3}{\infty}}\text{”} = \text{“}\frac{-\infty\cdot(1-0)}{-1+0}\text{”} = \text{“}\frac{-\infty\cdot 1}{-1}\text{”} = \text{“}\frac{-\infty}{-1}\text{”} = \infty$$

More in general we have:

$$\lim_{x\to\pm\infty} \frac{P(x)}{Q(x)} = \begin{cases} 0 & \text{if } \deg P < \deg Q \\ \text{ratio of the leading coefficients} & \text{if } \deg P = \deg Q \\ \infty \text{ or } -\infty & \text{if } \deg P > \deg Q \end{cases}$$

Indeed we have the following theorem:

Th: If $a_n, b_m \neq 0$ then $\displaystyle\lim_{x\to\pm\infty} \frac{a_n x^n + \dots + a_0}{b_m x^m + \dots + b_0} = \lim_{x\to\pm\infty} \frac{a_n x^n}{b_m x^m}$.

you can find the proof at the end of this PDF.

Asymptotes

Def: The line $x=a$ is a vertical asymptote of the curve $y=f(x)$ if $x=a$ is an infinite discontinuity, i.e. if one of the following is true

$$\lim_{x\to a} f(x) = \infty \text{ or } -\infty$$

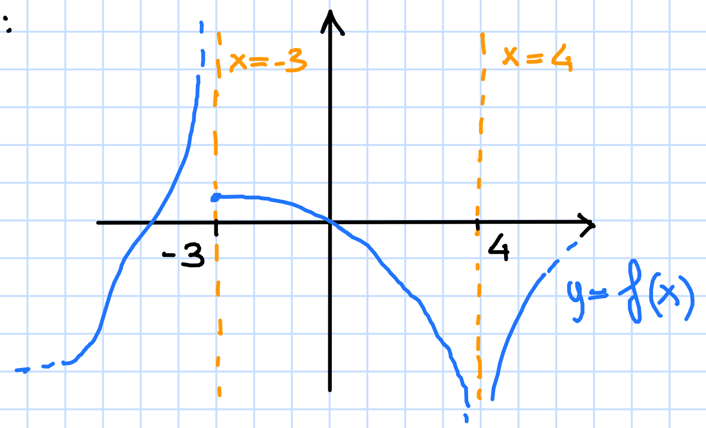
or

$$\lim_{x\to a^-} f(x) = \infty \text{ or } -\infty$$

or

$$\lim_{x\to a^+} f(x) = \infty \text{ or } -\infty$$

ex:

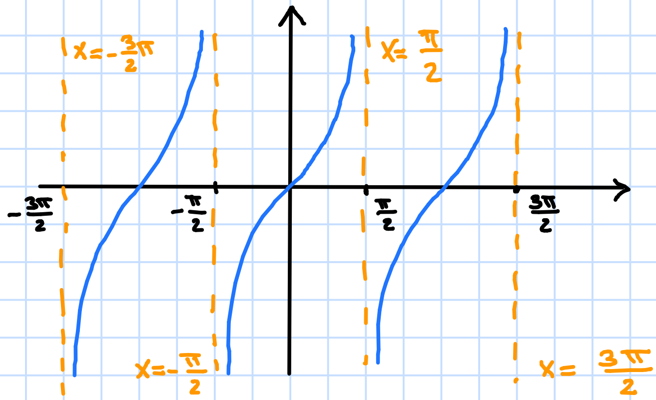


$x=-3$ and $x=4$ are two vertical asymptotes for f. Indeed:

$$\lim_{x\to -3^-} f(x) = \infty$$

$$\lim_{x\to 4} f(x) = -\infty$$

Remark: • A function can have infinitely many asymptotes. For instance the function $\tan(x)$ has a vertical asymptote at $x = \frac{\pi}{2} + k\pi$, for every integer k.



• The vertical asymptotes of a rational function have to be found in correspondence of the values that make the denominator 0.

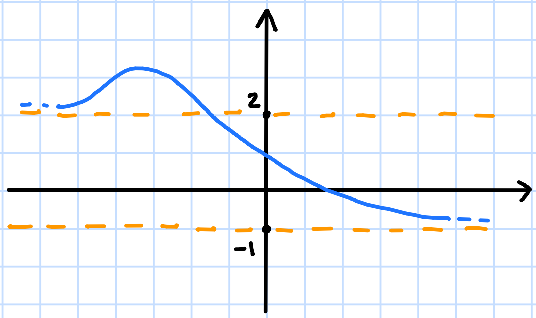
⚠ Warning: Not all the values that make the denominator 0 correspond to vertical asymptotes.

Indeed if $\frac{P(x)}{Q(x)}$ is a rational function and $Q(a)=0$, then $x=a$ can be an infinite or a removable discontinuity, and if a is a removable discontinuity then $x=a$ is not a vertical asymptote.

(An example of this fact is provided later).

Def: The line $y=L$ is a horizontal asymptote of the curve $y=f(x)$ if:

either $\lim_{x\to\infty} f(x) = L$ or $\lim_{x\to-\infty} f(x) = L$.



$y=2$ and $y=-1$ are two horizontal asymptotes since

$\lim_{x\to-\infty} f(x) = 2$ and $\lim_{x\to\infty} f(x) = -1$

Note that a function can cross its horizontal asymptotes.

Remark:
- A function f can have at most two different horizontal asymptotes, one at ∞ and one at $-\infty$.

  In particular f has exactly two different horizontal asymptotes if $\lim_{x\to\infty} f(x) = L_1$ and $\lim_{x\to-\infty} f(x) = L_2$, with $L_1 \neq L_2$.

- A constant function $f(x) = c$ has a horizontal asymptote of equation $y = c$.

Typical exercise about asymptotes

Write the equations of the vertical and horizontal asymptotes of the following rational function:

$$f(x) = \frac{x^2+6x+9}{x^2+2x-3}.$$

Solution

- HORIZONTAL ASYMPTOTE(S) $\longrightarrow$ Compute $\lim_{x\to\infty} f(x)$ and $\lim_{x\to-\infty} f(x)$

$$\lim_{x\to\infty} f(x) = \lim_{x\to\infty} \frac{x^2+6x+9}{x^2+2x-3} = \lim_{x\to\infty} \frac{\cancel{x^2}\left(1+\overset{\nearrow 0}{\frac{6}{x}}+\overset{\nearrow 0}{\frac{9}{x^2}}\right)}{\cancel{x^2}\left(1+\underset{\searrow 0}{\frac{2}{x}}-\underset{\searrow 0}{\frac{3}{x^2}}\right)} = \frac{1}{1} = 1$$

In the same way it is possible to show that $\lim_{x\to-\infty} f(x) = 1$.

Then $y = 1$ is the only horizontal asymptote for f.

recall that for a horizontal line it is the y-coordinate to be constant.

- VERTICAL ASYMPTOTE(S) $\longrightarrow$ Find the value(s) that make the denominator 0 and compute the limit when x approaches those values.

$$\text{denominator} = 0 \iff x^2+2x-3 = 0 \iff (x-1)(x+3) = 0$$
$$\iff x = 1 \quad \text{or} \quad x = -3.$$

Our candidates to be infinite discontinuities are $x = 1$ and $x = -3$.

We have:

- $\lim_{x\to 1^+} \frac{x^2+6x+9}{x^2+2x-3} = \lim_{x\to 1^+} \frac{(x+3)^{\cancel{2}}}{(x-1)\cancel{(x+3)}} = \text{"}\frac{1+3}{0^+}\text{"} = \text{"}\frac{4}{0^+}\text{"} = +\infty \Rightarrow$

$x>1 \Leftrightarrow x-1>0$

$\Rightarrow$ 1 is an infinite discontinuity and $x=1$ a vertical asymptote.

- $\lim_{x\to -3} \frac{x^2+6x+9}{x^2+2x-3} = \lim_{x\to -3} \frac{(x+3)^{\cancel{2}}}{(x-1)\cancel{(x+3)}} = \frac{-3+3}{-3-1} = \frac{0}{-4} = 0$

$\Rightarrow$ -3 is a removable discontinuity and <u>does not</u> correspond to a vertical asymptote.

<u>Conclusion</u>: f has a horizontal asymptote at $y=1$ and a vertical asymptote at $x=1$.

<u>EXERCISE</u>

Sketch the graph of a function f which satisfies simultaneously the following conditions:

$\lim_{x\to -\infty} f(x) = -2$, $\lim_{x\to 1^-} f(x) = \infty$, $f(1) = 2$,

$\lim_{x\to 1^+} f(x) = 2$, $\lim_{x\to \infty} f(x) = 1$.

<u>Solution</u>

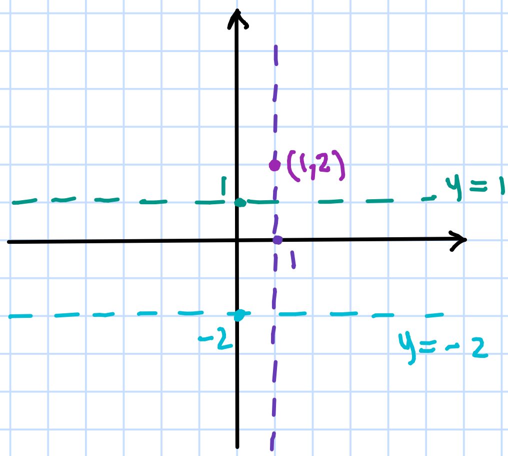
① Recognize and draw the horizontal / vertical asymptotes of the function and the points through which the graph passes.

$\lim_{x\to -\infty} f(x) = -2 \Rightarrow y=-2$ is a horiz. asymp.

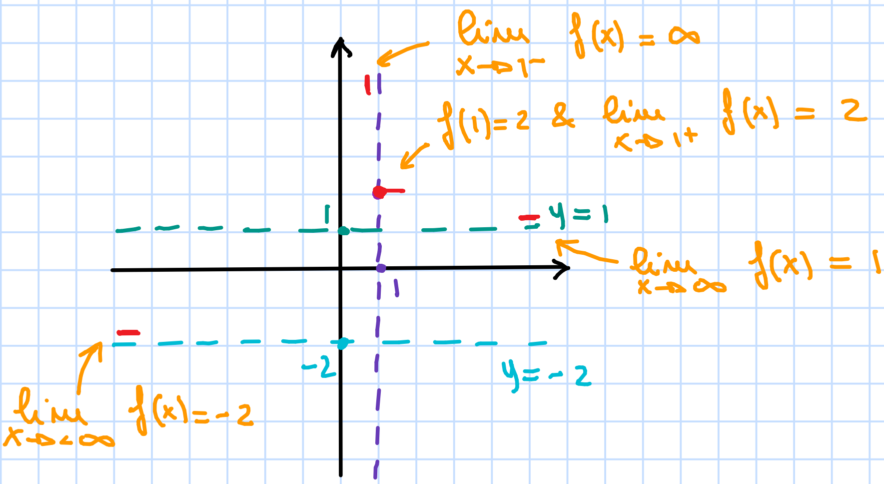
$\lim_{x\to \infty} f(x) = 1 \Rightarrow y=1$ is a horiz. asymp.

$\lim_{x\to 1^-} f(x) = \infty \Rightarrow x=1$ is a vertical asymp.

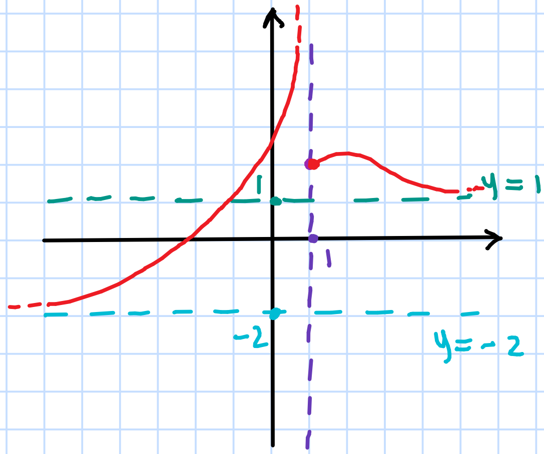
$f(1) = 2 \Rightarrow (1,2)$ is a point of the graph



② Write all the conditions as little marks — on your graph.



③ Connect the marks together (and make sure that your graph passes the vertical line test)



④ Check that your graph is one of the correct answers to the problem by verifying that it satisfies all the required conditions.

Calculating limits

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In the following tables the writing "$\lim\limits_{x\to\square} f(x)$" stands for $\lim\limits_{x\to a} f(x)$, $\lim\limits_{x\to a^-} f(x)$, $\lim\limits_{x\to a^+} f(x)$, $\lim\limits_{x\to\infty} f(x)$ or $\lim\limits_{x\to-\infty} f(x)$, L_1 and L_2 are real numbers (possibly equal to 0, unless otherwise specified) and the symbol ☠ means an *indeterminate form* (we recall that a form of limit is said to be *indeterminate* when knowing the limiting behavior of individual parts of the expression is not sufficient to actually determine the overall limit).

Sum

$\lim\limits_{x\to\square} f(x)$	$\lim\limits_{x\to\square} g(x)$	$\lim\limits_{x\to\square} f(x)+g(x)$
L_1	L_2	L_1+L_2
L_1	∞	∞
L_1	$-\infty$	$-\infty$
∞	L_2	∞
∞	∞	∞
∞	$-\infty$	☠
$-\infty$	L_2	$-\infty$
$-\infty$	∞	☠
$-\infty$	$-\infty$	$-\infty$

We can consider the limit of the difference of two functions as the limit of a sum in the following way:

$$\lim_{x\to\square} f(x)-g(x) = \lim_{x\to\square} f(x)+(-g(x)).$$

Hence, for example, if $\lim\limits_{x\to\square} f(x)=\infty$ and $\lim\limits_{x\to\square} g(x)=-\infty$ we have $\lim\limits_{x\to\square} f(x)-g(x) = $ "$\infty-(-\infty)$" $=$ "$\infty+\infty$" $=\infty$.

Examples.

1) $\lim\limits_{x\to-\infty} \sqrt{3-4x}-x+1 = \lim\limits_{x\to-\infty}(\sqrt{3-4x}) + \lim\limits_{x\to-\infty}(-x) + \lim\limits_{x\to-\infty} 1 =$ "$\infty+\infty-1$" $=\infty$.

2) $\lim\limits_{x\to\infty} x^2-x = \lim\limits_{x\to\infty}(x^2) + \lim\limits_{x\to\infty}(-x) =$ "$\infty-\infty$" $\to$ ☠ (look at the examples of the product for seeing how to escape to the indeterminate form...)

PRODUCT

$\lim_{x\to\square} f(x)$	$\lim_{x\to\square} g(x)$	$\lim_{x\to\square} f(x)g(x)$
L_1	L_2	$L_1 \cdot L_2$
$L_1 > 0$	∞	∞
$L_1 > 0$	$-\infty$	$-\infty$
0	∞	☠
0	$-\infty$	☠
$L_1 < 0$	∞	$-\infty$
$L_1 < 0$	$-\infty$	∞
∞	∞	∞
∞	$-\infty$	$-\infty$

The table for the product can be completed by using the commutative property of the product (that is the reason why in the table for example the case $\lim_{x\to\square} f(x) = \infty$ and $\lim_{x\to\square} g(x) = L_2$ does not appear).

Moreover we can deduce the table for the limit of the quotient of two functions by considering the quotient as a product:

$$\lim_{x\to\square} \frac{f(x)}{g(x)} = \lim_{x\to\square} f(x) \cdot \frac{1}{g(x)}$$

and using the following table:

$\lim_{x\to\square} g(x)$	$\lim_{x\to\square} \frac{1}{g(x)}$
L	$\frac{1}{L}$
$0^+ \ (>0)$	∞
$0^- \ (<0)$	$-\infty$
∞	0

We deduce that also $\dfrac{0}{0}$ and $\dfrac{\infty}{\infty}$ are indeterminate forms ☠.

Examples.

1) $\lim_{x\to\infty} x^2 - x = \lim_{x\to\infty} x(x-1) = \lim_{x\to\infty} x \cdot \lim_{x\to\infty} (x-1) = \text{“}\infty \cdot \infty\text{”} = \infty$

2) $\lim_{x\to\frac{\pi}{2}^+} \frac{1}{\cos x} + \frac{1}{\frac{\pi}{2} - x} = \text{“}\frac{1}{0^-} + \frac{1}{0^-}\text{”} = \text{“} - \infty - \infty\text{”} = -\infty.$

The case of rational functions

We recall that a rational function is a function of the form:

$$f(x) = \frac{P(x)}{Q(x)} = \frac{a_n x^n + a_{n-1}x^{n-1} + \cdots + a_0}{b_m x^m + b_{m-1}x^{m-1} + \cdots + b_0}$$

where $P(x)$ and $Q(x)$ are two polynomials with real coefficients of degree n and m respectively ($a_n \neq 0, b_m \neq 0$).

We consider here the particular limits

$$\lim_{x\to\infty} \frac{P(x)}{Q(x)} \qquad \text{or} \qquad \lim_{x\to-\infty} \frac{P(x)}{Q(x)}.$$

Theorem 1. *We have:*

$$\lim_{x\to\pm\infty} \frac{a_n x^n + a_{n-1}x^{n-1} + \cdots + a_0}{b_m x^m + b_{m-1}x^{m-1} + \cdots + b_0} = \lim_{x\to\pm\infty} \frac{a_n x^n}{b_m x^m}$$

Proof.

$$\begin{aligned}
\lim_{x\to\pm\infty} \frac{a_n x^n + a_{n-1}x^{n-1} + \cdots + a_0}{b_m x^m + b_{m-1}x^{m-1} + \cdots + b_0} &= \lim_{x\to\infty} \frac{x^n(a_n + a_{n-1}\frac{1}{x} + \cdots + a_0\frac{1}{x^n})}{x^m(b_m + b_{m-1}\frac{1}{x^{m-1}} + \cdots + b_0\frac{1}{x^m})} = \\
&= \lim_{x\to\pm\infty} \frac{x^n}{x^m} \cdot \lim_{x\to\pm\infty} \frac{a_n + a_{n-1}\frac{1}{x} + \cdots + a_0\frac{1}{x^n}}{b_m + b_{m-1}\frac{1}{x^{m-1}} + \cdots + b_0\frac{1}{x^m}} = \\
&= \left(\lim_{x\to\pm\infty} \frac{x^n}{x^m}\right) \cdot \frac{a_n + 0 + \cdots + 0}{b_m + 0 + \cdots + 0} = \\
&= \lim_{x\to\pm\infty} \frac{a_n x^n}{b_m x^m}.
\end{aligned}$$

$\square$

Hence the limit takes different values according to different cases:

1) $n > m$

$$\lim_{x\to\infty} \frac{a_n x^n + a_{n-1}x^{n-1} + \cdots + a_0}{b_m x^m + b_{m-1}x^{m-1} + \cdots + b_0} = \lim_{x\to\infty} \frac{a_n x^n}{b_m x^m} = \frac{a_n}{b_m} \lim_{x\to\infty} x^{n-m} = \begin{cases} \infty, & \text{if } \frac{a_n}{b_m} > 0 \\ -\infty, & \text{if } \frac{a_n}{b_m} < 0 \end{cases}.$$

$$\lim_{x\to-\infty} \frac{a_n x^n + a_{n-1}x^{n-1} + \cdots + a_0}{b_m x^m + b_{m-1}x^{m-1} + \cdots + b_0} = \lim_{x\to-\infty} \frac{a_n x^n}{b_m x^m} = \frac{a_n}{b_m} \lim_{x\to-\infty} x^{n-m} = \begin{cases} \infty, & \text{if } \frac{a_n}{b_m} > 0 \text{ and } n-m \text{ even} \\ -\infty, & \text{if } \frac{a_n}{b_m} > 0 \text{ and } n-m \text{ odd} \\ -\infty, & \text{if } \frac{a_n}{b_m} < 0 \text{ and } n-m \text{ even} \\ \infty, & \text{if } \frac{a_n}{b_m} < 0 \text{ and } n-m \text{ odd} \end{cases}.$$

2) $n = m$

$$\lim_{x\to\pm\infty} \frac{a_n x^n + a_{n-1}x^{n-1} + \cdots + a_0}{b_n x^n + b_{n-1}x^{n-1} + \cdots + b_0} = \lim_{x\to\pm\infty} \frac{a_n x^n}{b_n x^n} = \frac{a_n}{b_n}.$$

3) $n < m$

$$\lim_{x\to\pm\infty} \frac{a_n x^n + a_{n-1}x^{n-1} + \cdots + a_0}{b_m x^m + b_{m-1}x^{m-1} + \cdots + b_0} = \lim_{x\to\pm\infty} \frac{a_n x^n}{b_m x^m} = \frac{a_n}{b_m} \lim_{x\to\pm\infty} \frac{1}{x^{m-n}} = 0.$$

Examples.

1) $\lim\limits_{x\to\infty} \dfrac{3x^2 - x + 5}{4x^2 - 1} = \lim\limits_{x\to\infty} \dfrac{x^2(3 - \frac{1}{x} + \frac{5}{x^2})}{x^2(4 - \frac{1}{x^2})} = \lim\limits_{x\to\infty} \dfrac{3 - \frac{1}{x} + \frac{5}{x^2}}{4 - \frac{1}{x^2}} = \dfrac{3}{4}$

2) $\lim\limits_{x\to-\infty} \dfrac{3x^4 - 2x^2 + 1}{-2x^2 - 2} = \lim\limits_{x\to-\infty} \dfrac{x^4(3 - 2\frac{1}{x^2} + \frac{1}{x^4})}{x^2(-2 - \frac{2}{x^2})} = \lim\limits_{x\to-\infty} \dfrac{x^2(3 - 2\frac{1}{x^2} + \frac{1}{x^4})}{-2 - \frac{2}{x^2}} = \text{“}\dfrac{\infty \cdot 3}{-2}\text{”} = -\infty$