LIMIT OF A FUNCTION (Sec. 1.3 of the book)

The concept of limit of a function is very, very very important, since all calculus is based upon it.

Indeed, we will see that the idea of limit, other than being important in itself, is also behind the concepts of derivative and integral.

$$\boxed{\text{LIMIT} \quad \lim_{x \to a} f(x)}$$

DERIVATIVE

$$\frac{d}{dx} f(x)$$

INTEGRAL

$$\int f(x)\, dx$$

The limit of a function concerns the behaviour of that function near a particular input.
In some sense it is the prediction of the value of a function we should get at a point.

<u>ex</u>: Let us go back to a previous example:

$$f(x) = \frac{x^2+2x}{x} = \frac{x(x+2)}{x}$$

For all real numbers we can compute the value of the function, except 0. Indeed:

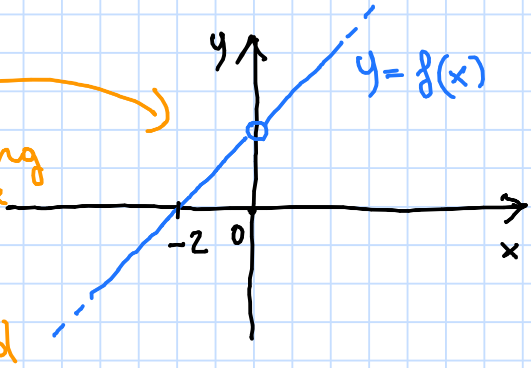
$f(0) = $ "$\frac{0}{0}$" which is undefined (which implies that 0 is not in the domain of f)

But, what is the "behaviour" of f "near" 0, that is when my input is really close to 0 $(-0.001, -0.00001, 0.000000001, \dots)$?

This is why we need limits! (as we saw in the introduction with the istantaneous velocity, where we also met with $\frac{0}{0}$)

We saw that $f(x) = x+2$ for all $x \neq 0$ and has the following graph:

we can not see what is actually happening at x=0, because there is a hole, but we can see what it is happening around

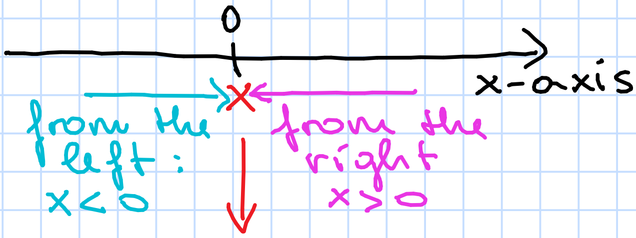


We introduce the following notation:

$\lim_{x \to 0} f(x) = ?$ and we read: The limit as x approaches 0 of f(x) equals ...

this means that we are approaching 0 on the x-axis, and we can approach it from the left and from the right

here we are looking for a y-value which is a real number



but we do not care about what it is going on at x=0 !!!

When we write $\lim_{x \to 0} f(x)$ we have automatically to think that we have to check what it is happening in both sides of 0, from the left and from the right

The most intuitive way to find this limit is to build a table of values or better two tables of values (one from the left with x<0, and one from the right, with x>0) that approach more and more 0:

from the left
$x < 0$

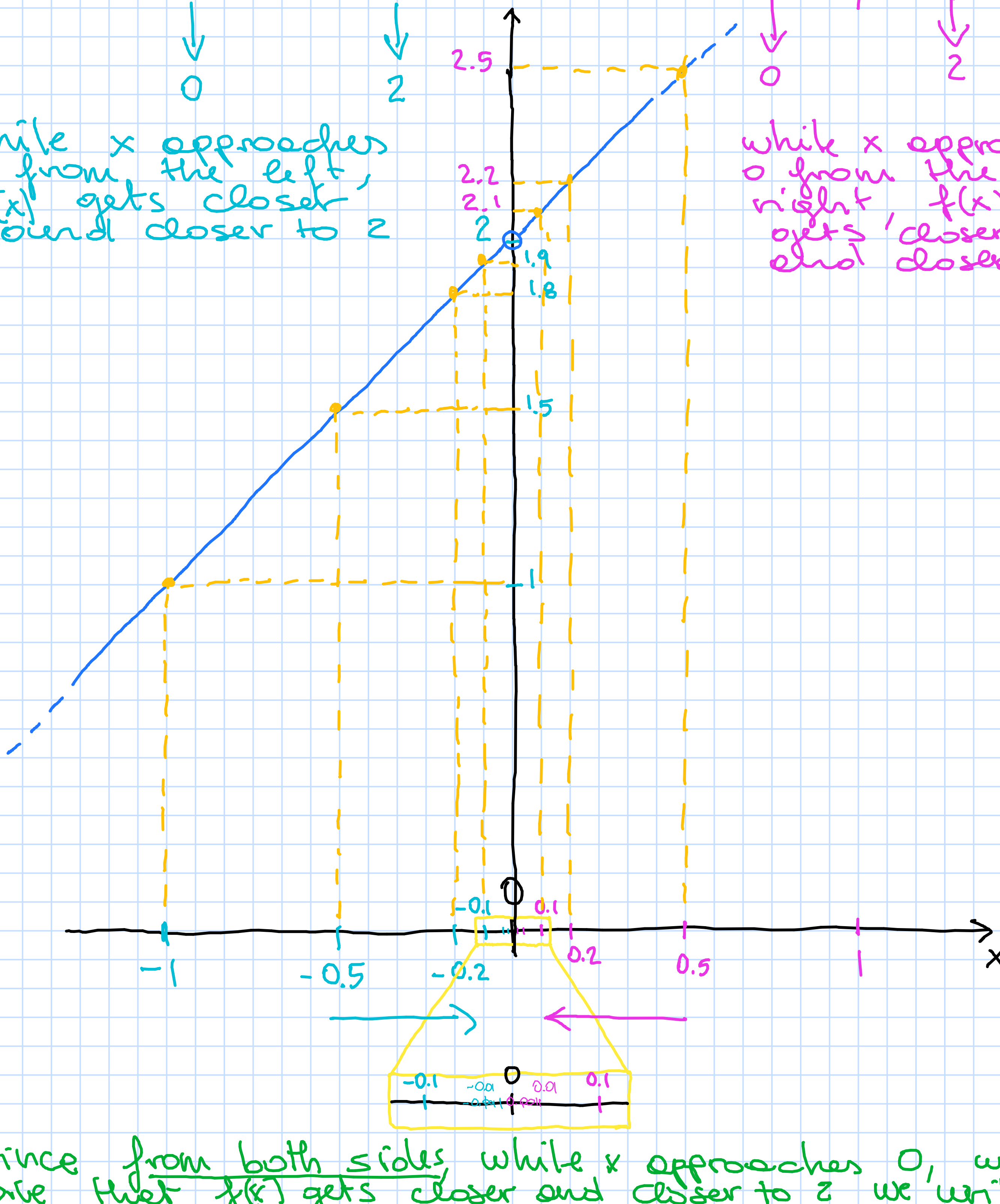
x	$f(x)$
-1	1
-0.5	1.5
-0.2	1.8
-0.1	1.9
-0.01	1.99
-0.001	1.999
-0.000001	1.999999
$\downarrow$	$\downarrow$
0	2

while x approaches 0 from the left, f(x) gets closer and closer to 2

from the right
$x > 0$

x	$f(x)$
1	3
0.5	2.5
0.2	2.2
0.1	2.1
0.01	2.01
0.001	2.001
0.000001	2.000001
$\downarrow$	$\downarrow$
0	2

while x approaches 0 from the right, f(x) gets closer and closer to 2

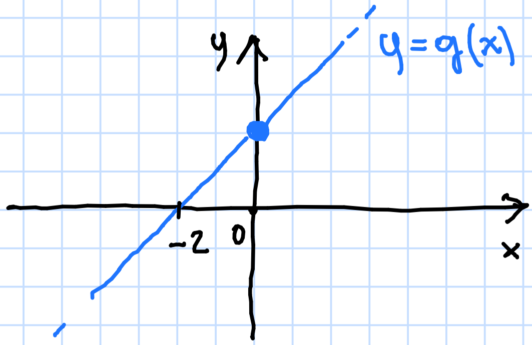


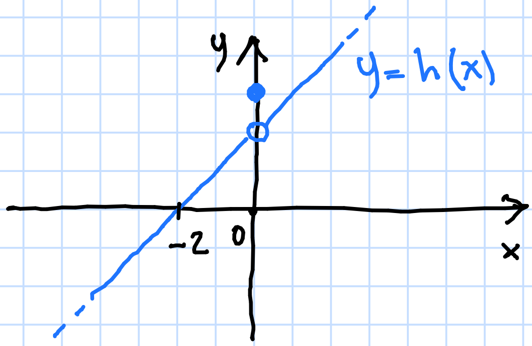
Since from both sides while x approaches 0, we have that f(x) gets closer and closer to 2 we write

$$\lim_{x \to 0} f(x) = 2$$

Question: And what about if at 0 the function was defined to be 2 or another value, that is if we are in one of the following two situations?

$$g(x) = x+2 \qquad h(x) = \begin{cases} \dfrac{x(x+2)}{x} & \text{if } x \neq 0 \\ 3 & \text{if } x = 0 \end{cases}$$





This does not change anything!

We have also $\lim_{x \to 0} g(x) = 2$ and $\lim_{x \to 0} h(x) = 2$

Indeed when we compute limits, we do not care about what the function is doing exactly at 2.

We only care about what is happening just around!

We have the following definition:

for a more formal and precise definition look inside the book (ε, δ definition)

Def: Let f be a function and a a real number. Suppose that f is defined in a neighbourhood of a (this means in some open interval that contains a except possibly at a itself).

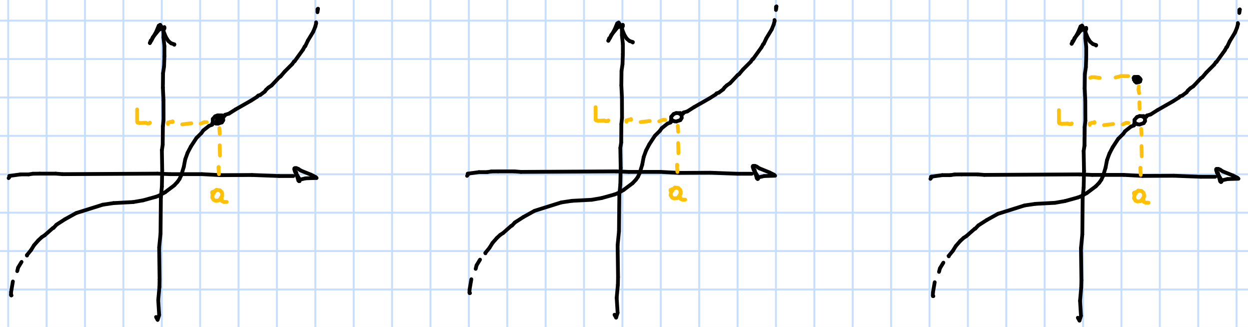
We write

$\lim_{x \to a} f(x) = L$: "the limit of $f(x)$ as x approaches a equals L"

if we can make the values of $f(x)$ arbitrary closed to L by taking x to be sufficiently close to a (on either side of a) but not equal to a.

Again, that means that in finding the limit we never consider $x = a$

So, in either of the following cases:



$f(a) = L$ — a not in the domain of f — $f(a) \neq L$

we have

$$\lim_{x \to a} f(x) = L$$

Sometimes left-hand and right-hand limits are not the same.

This leads us to consider one-sided limits.

We call and denote them:

→ left-hand limit : $\lim_{x \to a^-} f(x)$

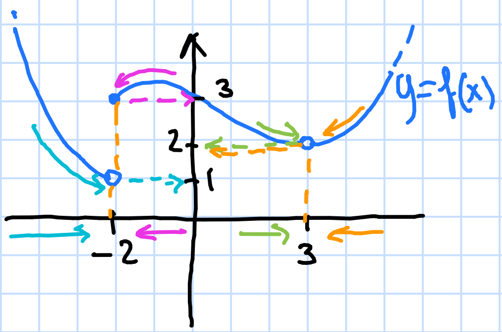
this minus here means that we are considering inputs less than a : $x < a$

→ right-hand limit : $\lim_{x \to a^+} f(x)$

this plus here means that we are considering inputs greater than a: $x > a$

Let us consider the following example:

ex :



$\lim_{x \to -2^-} f(x) = 1$ $\qquad$ $\lim_{x \to 3^-} f(x) = 2$

$\lim_{x \to -2^+} f(x) = 3$ $\qquad$ $\lim_{x \to 3^+} f(x) = 2$

Now:

different left-hand and right-hand limit

$$\lim_{x\to -2^-} f(x) \neq \lim_{x\to 2^+} f(x)$$

$$\Downarrow$$

$\lim_{x\to -2} f(x)$ does not exist (we also write DNE)

same left-hand and right-hand limit

$$\lim_{x\to 3^-} f(x) = \lim_{x\to 3^+} f(x) = 2$$

$$\Downarrow$$

$$\lim_{x\to 3} f(x) = 2$$

Indeed the overall limit exists if and only if the left-hand and the right hand limit exist and are equal.

More precisely:

$$\lim_{x\to a} f(x) = L \iff \lim_{x\to a^-} f(x) = \lim_{x\to a^+} f(x) = L$$

Let us consider another example:

$$f(x) = \frac{1}{x^2}.$$

The domain of f is $\mathbb{R}\setminus\{0\}$ (or $(-\infty,0)\cup(0,\infty)$).

Hence it is interesting to compute:

$$\lim_{x\to 0} \frac{1}{x^2}$$

Let us build again a table of value. We remark that f is an even function:

indeed $\quad f(-x) = \frac{1}{(-x)^2} = \frac{1}{x^2} = f(x)$ (you can find the def. of an even function in Class 1)

This implies that f takes the same value at opposite numbers:

x	$f(x)$
± 1	1
± 0.5	4
± 0.2	25
± 0.1	100
± 0.05	400
± 0.01	10,000
± 0.001	1,000,000
$\downarrow$ 0	

We remark that while x approaches 0, $f(x)$ becomes arbitraly large.

So $\lim_{x\to 0} \frac{1}{x^2}$ does not exist (because we are not getting closer to a specific number), but it does not exist in a particular way...

we will come back to this example in Sec. 1.6