★ Class 1 : 01/08/2018

They say that Mathematics is an international language, but... handwriting can be different from country to country!

ex: My numbers:

0, 1 or 1, 2, 3, 4, 5, 6, 7, 8, 9

not to be confused with the letter "g"

Notation

It can be usefull to know the following notation:

- $\mathbb{N} = \{0, 1, 2, 3, \ldots\}$ is the set of natural integers.
- $\mathbb{Z} = \{\ldots, -3, -2, -1, 0, 1, 2, 3, \ldots\}$ is the set of integers (with sign).
- $\mathbb{Q} = \{\frac{a}{b}$, where a is in $\mathbb{Z}$ and $b \neq 0$ is in $\mathbb{Z}\}$ is the set of rational numbers (fractions).
- $\mathbb{R}$ is the set of real numbers (rational + irrational numbers).

- $\in$ = "belongs to" : it is used for saying that an element belongs to a set
- $\notin$ = "does not belong to"

ex: $\pi \in \mathbb{R}$ but $\pi \notin \mathbb{Q}$

pi, the constant of the circle ~ 3.14

π is a real number, but not a rational number (π is an irrational number)

- $\subseteq$ = "is contained in": it is used for saying that a set is contained in another set.
- $\supseteq$ = "contains"
- $\not\subseteq$ = "is not contained"
- $\not\supseteq$ = "does not contain"

ex: $\mathbb{N} \subseteq \mathbb{Z}$: $\mathbb{N}$ is contained in $\mathbb{Z}$ because all natural numbers are integers

$\mathbb{R} \supseteq \mathbb{Q}$

INTRODUCTION TO CALCULUS

- Etimology

The word "calculus" comes from Latin and means "a small pebble or stone used for counting".

- A little bit of history

Modern calculus was developed in 17th-century by Newton and Leibniz indipendently of each other (even if there was a lot of controversy... mathematicians can be very jealous of their results!)

Newton: first to apply calculus to general physics.

Leibniz: developed much of the notation used in calculus today.

- What does calculus study?

Calculus is the study of change and it studies change by studying "istantaneous" change (over very small interval of time).

Let us try to understand this with an easy example: the motion of an object.

An example: the motion of an object along a fixed path

- Let us fix a point on the path. At any time we can describe the position (= "distance" from the fixed point) of the object.

  We can say that the motion of an object is characterized by the set of its numerical positions at rele-vant points in time.

  This is what we usually call a "function", which is one of the basic notions of calculus!

- What does it change in this example?
  The position ("distance" of the object from a fixed point) varies with time

- And how does the position change with time?
  This depends on the velocity of the object...

AVERAGE VELOCITY VS ...

Sam and Alex are traveling in the car ... but the speedometer is broken.

$$\frac{\text{displacement}}{\text{time}} = \frac{1.2\ \text{km}}{1\ \text{min}} = \frac{1.2\ \text{km}}{\frac{1}{60}\ \text{h}} = 1.2\ \text{km} \cdot 60\ \frac{1}{\text{h}} = 72\ \frac{\text{km}}{\text{h}}$$

Alex: "Hey Sam! How fast are we going now?"

Sam: "Wait a minute ..."

"Well in the last minute we went 1,2 km, so we are going:"

1,2 km per minute x 60 minutes in an hour = **72 km/h**

Alex: "No, Sam! Not our **average** for the last minute, or even the last second, I want to know our speed RIGHT NOW."

... ISTANTANEOUS VELOCITY

Sam: "OK, let us measure it up here ... at this road sign... NOW!"

here we need LIMITS!!

"OK, we were AT the sign for **zero seconds**, and the distance was ... **zero meters**!"

The speed is 0m / 0s = 0/0 = **I Don't Know**!

"I can't calculate it Sam! I need to know **some** distance over **some** time, and you are saying the time should be zero? Can't be done."

source: www.mathisfun.com

Two problems

1) Find the istantaneous velocity (called more in general the derivative of the function)

↓

DIFFERENTIAL CALCULUS

2) Find the position by knowing the istantaneous velocity at all time (or more in general, find the function by knowing its derivative)

↓

INTEGRAL CALCULUS

CALCULUS

differential calculus

integral calculus

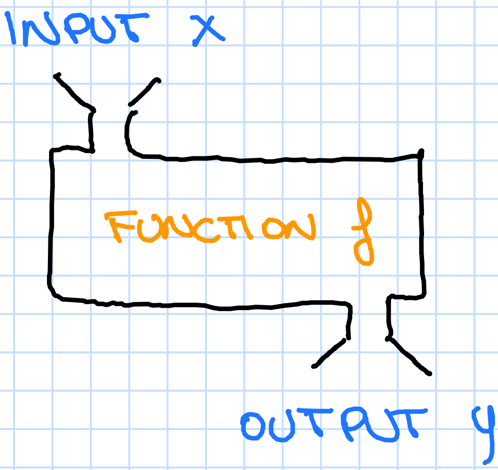
this is our goal!

related by the FUNDAMENTAL THEOREM OF CALCULUS

First notion : FUNCTION
(This is section 1.1 of the book : "Functions and their representations")

- **What is a function?**

We can imagine a function as a machine or a black box that for each input x returns a single output y.



DOMAIN = set of all possible inputs
RANGE = set of all possible outputs

More formally:

Def : A function f is a rule that assigns to each element x in a set D exactly one element called $f(x)$ in a set E

$$f : D \longrightarrow E$$
$$x \longmapsto f(x)$$

Remarks: • Usually we consider D and E to be subset of the set of real numbers $\mathbb{R}$.

• The range of f is contained in E, but it is possibly different.

How to represents a function

1) verbally (in words)

bla bla bla

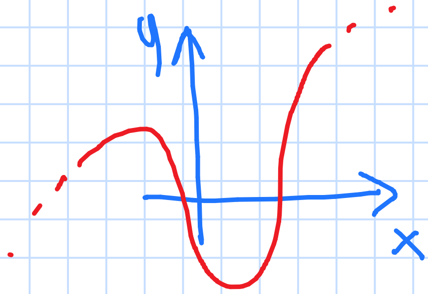
2) algebraically (explicit formula)

$$f(x) = \sin(x^2) + x\cos(x)$$

3) numerically (table of values)

x	$f(x)$
1	0
2	3
3	−1
π	0
.	.
⋮	⋮

4) visually (graph)



example

1) In words

Let us consider the function that associates to each real number its square

2) Algebraically

$$f: \mathbb{R} \longrightarrow \mathbb{R}$$
$$x \longmapsto x^2$$

or $f(x) = x^2$

domain: $D = \mathbb{R}$

or I can write

$D = (-\infty, \infty)$

Indeed for each real number I can compute its square (so each real number is an input for my function f)

range $= [0, +\infty)$

Indeed the square of each real number is non-negative. In formulas:

for all $x \in \mathbb{R}$, $x^2 \geq 0$.

3) Table of values

input → output

x	$f(x) = x^2$
-3	9
-2	4
-1	1
0	0
1	1
2	2
3	3

4) Geometrically

Def: The graph of a function f is the set of points of the plane of the form $(x, f(x))$, where x is in the domain

x-coordinate   y-coordinate

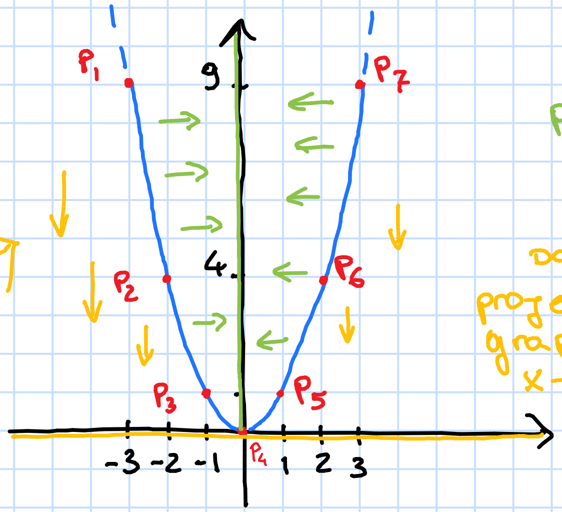
Remark: The curve of the graph of a function f has cartesian equation:

$$y = f(x)$$

If $f(x) = x^2$, we have from the previous table of values that:

P_1 $(-3, 9)$ x $f(x)$
P_2 $(-2, 4)$
P_3 $(-1, 1)$
P_4 $(0, 0)$
P_5 $(1, 1)$
P_6 $(2, 4)$
P_7 $(3, 9)$

are points on the graph of f.



RANGE
projection of the graph on the y-axis: $[0, \infty)$

DOMAIN
projection of the graph on the x-axis: $(-\infty, \infty)$

Also, from a geometrical point of view, we have that:

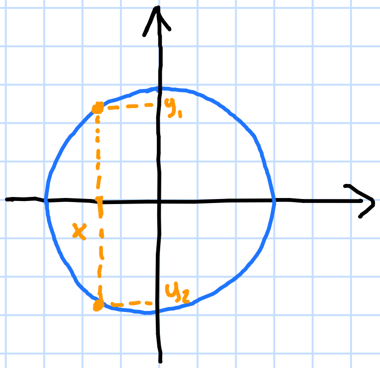
domain = projection of the graph of the function on the x-axis

range = projection of the graph of the function on the y-axis

Question: Are all the curves in the plane graphs of some function?

No!

ex: CIRCLE



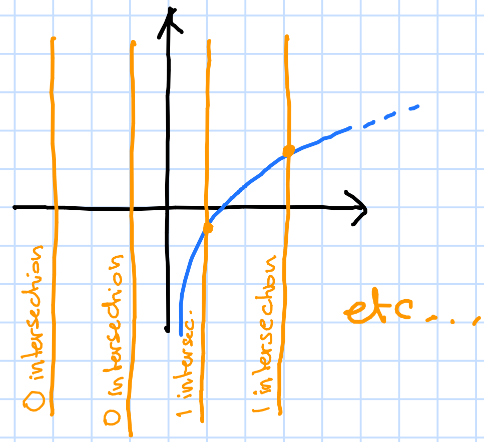
The circle can not be the graph of a function since, if it was the case, the input x would have two output y_1 and y_2.

VERTICAL LINE TEST

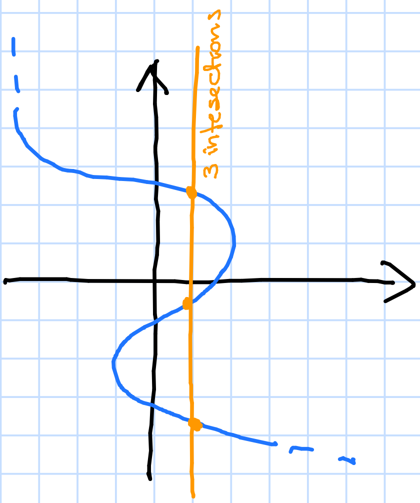
A curve in the plane is the graph of a function if and only if no vertical line intersects the curve more than one

(⚠ it can also have zero intersections)

ex:



By the vertical line test this is the graph of a function, since all the vertical lines have at most one intersection with the curve of the graph



By the vertical line test this is not the graph of a function, since there is a vertical line that has more than one intersection with the curve of the graph

EXERCISE : Find the domain of the following function.

$$f: \mathbb{R} \to \mathbb{R}, \quad f(x) = \frac{x^2+2x}{x}$$

Solution

Note : $f(x)$ is called a rational function since it is the quotient of two polynomials :

x^2+2x , x

numerator   denominator

Now, the domain of a rational function is given by the set of all real numbers that do not make the denominator equal to zero.

In our case :

$D = \{x \in \mathbb{R}$ such that $x \neq 0\}$

denominator

$\Rightarrow \quad D = (-\infty, 0) \cup (0, \infty)$

or I can write

$D = \mathbb{R} \setminus \{0\}$

difference of sets : this means all real numbers except 0.

Now let us consider $g(x) = x+2$.

Question : Is it $g = f$?

This is a good question, since we can rewrite

$$f(x) = \frac{x^2+2x}{x} = \frac{x(x+2)}{x}$$

and if now we simplify without thinking we get :

$$f(x) = \frac{\cancel{x}(x+2)}{\cancel{x}} = x+2$$

But we have to be careful, since we can not divide by 0!

So this simplification works for all $x \neq 0$.

Hence we get:

$$f(x) = g(x) \quad \text{for all} \quad x \neq 0.$$

Actually we have the following definition

Def: Two functions f and g are equal if and only if they have the same domain D and $f(x) = g(x)$ for all x in D:

$f = g \iff f$ and g have the same domain D and $f(x) = g(x)$ for all $x \in D$.

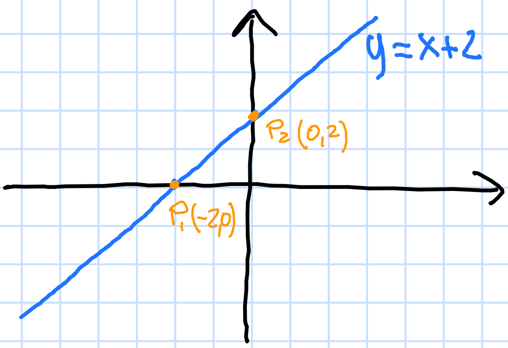
this symbol means "if and only if"

Here we are saying that in order to be equal f and g have to take the same values at each point of the domain.

So in our case, since $f(x) = \dfrac{x(x+2)}{x}$ has domain $D_f = \mathbb{R} \setminus \{0\}$ and $g = x+2$ has domain $D_g = \mathbb{R}$ and $D_f \neq D_g$, then $f \neq g$!

Hence the answer to the question is No!

Anyway we got that $f(x) = g(x)$ for all $x \neq 0$. Now the graph of g is a line:

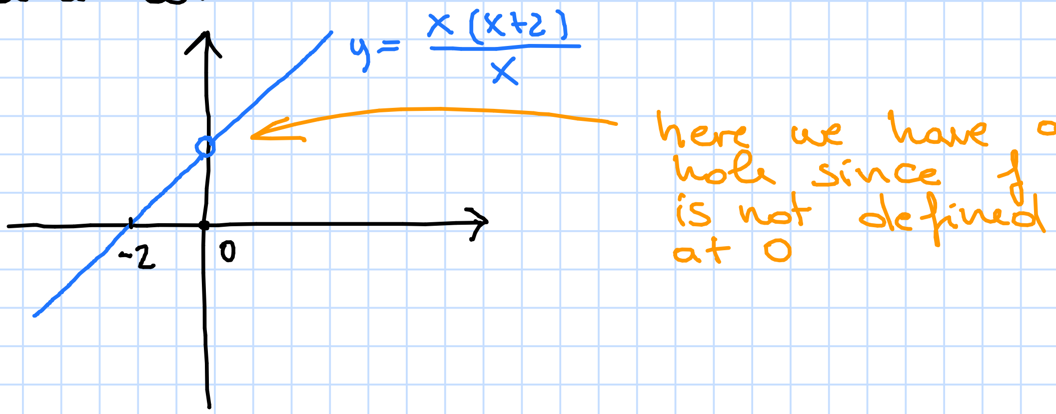


Recall that for drawing a line you need two points. Now $g(x) = x+2$, then:

$g(-2) = 0 \Rightarrow P_1(-2, 0)$

$g(0) = 2 \Rightarrow P_2(0, 2)$

Then, since $f(x) = g(x)$ for all $x \neq 0$ and f is not defined at $x=0$ (0 is not in the domain) we can easily draw also the graph of f, which will be a line with a hole:



EXERCISE: Find the domain of the following function:

$$f: \mathbb{R} \to \mathbb{R}, \quad f(x) = \sqrt{x+3}$$

Solution

The function f is called a root function.
Now a root function is defined if and only if the quantity "under the root" is greater or equal than 0.

In our case:

"if and only if" quantity under the root

$$f(x) = \sqrt{x+3} \text{ is defined} \iff x+3 \geq 0 \iff$$

$$\iff x+3-3 \geq 0-3 \iff x \geq -3 \iff x \in [-3, \infty)$$

belongs to

Hence, in this case $D = [-3, \infty)$

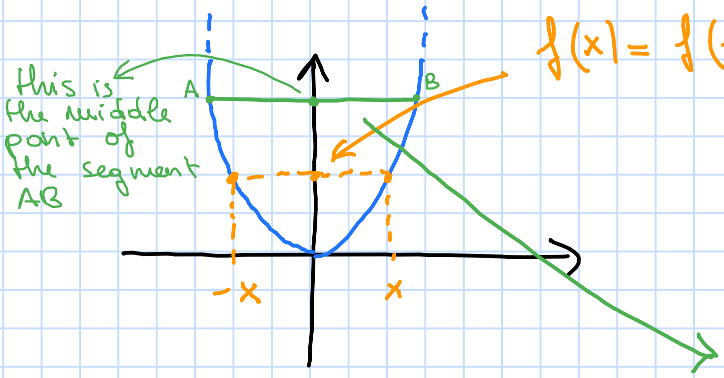
This part was not covered in class but it is good to be known.

ODD / EVEN FUNCTIONS

Odd / even functions are functions with a specific property:

Def: A function f with domain D is even if $f(x) = f(-x)$ for all $x \in D$.

ex: $f(x) = x^2$
This function is even since:
$$f(-x) = (-x)^2 = x^2 = f(x).$$



$f(x) = f(-x)$: the function takes the same values at x and $-x$.

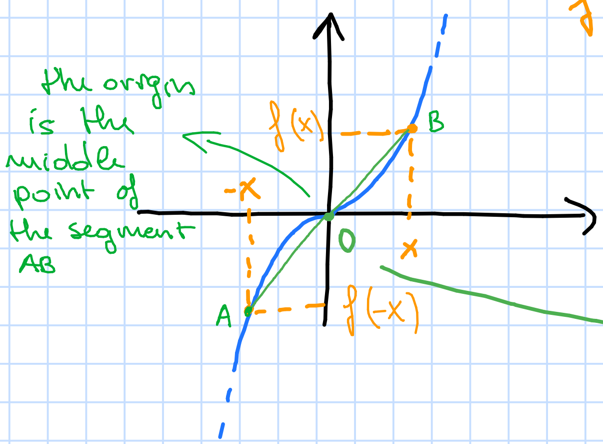
This property reflects in the graph in the fact that:

the graph of an even function is symmetric about the y-axis

Other examples: $f(x) = |x|$, $f(x) = \cos(x)$

Def: A function f with domain D is odd if $f(x) = -f(-x)$ for all $x \in D$.

ex: $f(x) = x^3$
This function is even since:
$$f(-x) = (-x)^3 = -x^3 = -f(x)$$



$f(x) = -f(-x)$: the function takes opposite values at x and $-x$.

This property reflects in the graph in the fact that:

the graph of an odd function is symmetric with respect to the origin $O\ (0,0)$

Other examples : $f(x) = \sin(x)$

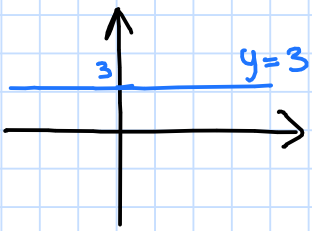
This part was not covered in class but it is good to be known (Sec. 1.2 of the book)

ESSENTIAL FUNCTIONS

- constant functions

$f(x) = c$, where $c \in \mathbb{R}$ is a constant.

ex: $f(x) = 3$

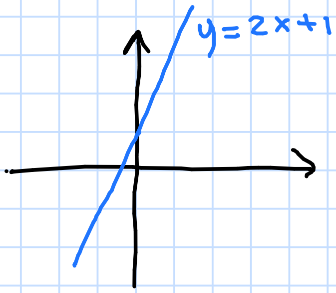


The graph of a constant function is a horizontal line since at each point the function takes the same value.

- linear functions

$f(x) = ax+b$, where $a, b \in \mathbb{R}$

ex: $f(x) = 2x+1$

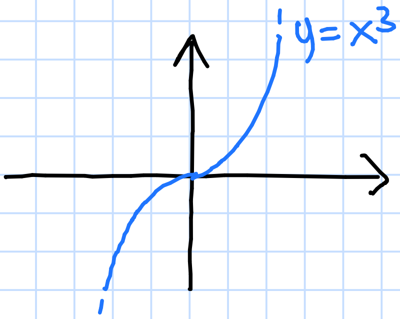


the graph of a linear function is a line.

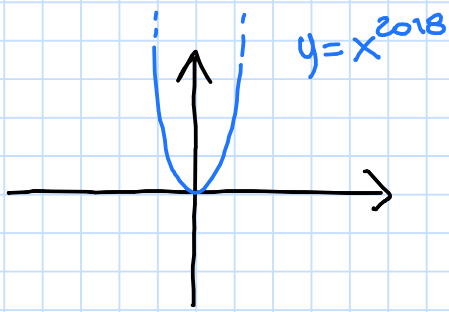
- power functions

$f(x) = x^n$, where $n \in \mathbb{N}$ (is a natural number)

ex: $f(x) = x^3$, $f(x) = x^{2018}$



odd function (every time that the exponent is odd)



even function (each time that the exponent is even)

4) polynomial

$f(x) = a_n x^n + \cdots + a_1 x + a_0$, where $a_0, a_1, \dots, a_n \in \mathbb{R}$

general form

"then"

A polynomial is defined for all $x \in \mathbb{R} \Rightarrow D = \mathbb{R}$

5) absolute value

$$f(x) = |x| = \begin{cases} x & \text{if } x \geq 0 \quad 1° \text{ case} \\ -x & \text{if } x < 0 \quad 2° \text{ case} \end{cases}$$

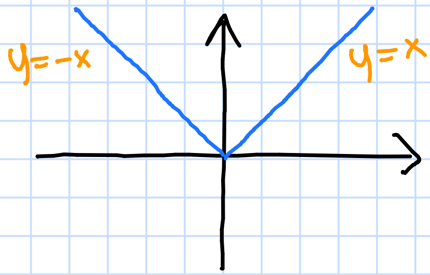
this is an example of a piecewise function

ex.: $|1| = 1$, $|-3| = -(-3) = 3$

1° case $1 \geq 0$ 2° case $-3 < 0$

that is, the absolute value of a real number is the real number without sign.

This implies $|x| \geq 0$ for all $x \in \mathbb{R}$



The graph is the union of two semi-lines.

Indeed we have the line $y = -x$ when $x < 0$ and $y = x$ when $x \geq 0$

This is also an example of even function

$D = \mathbb{R}$

6) Rational functions

$f(x) = \dfrac{P(x)}{Q(x)}$ where $P(x)$ and $Q(x)$ are polynomials

ex. $f(x) = \dfrac{x^2 + 2x - 1}{x^3 - 1}$

D: $x^3 - 1 \neq 0 \Rightarrow x^3 \neq 1$
$\Rightarrow x \neq 1 \Rightarrow D = \mathbb{R} \setminus \{1\}$

The domain of a rational function is the set of all real numbers that do not make the denominator equal to zero.

7) trigonometric functions

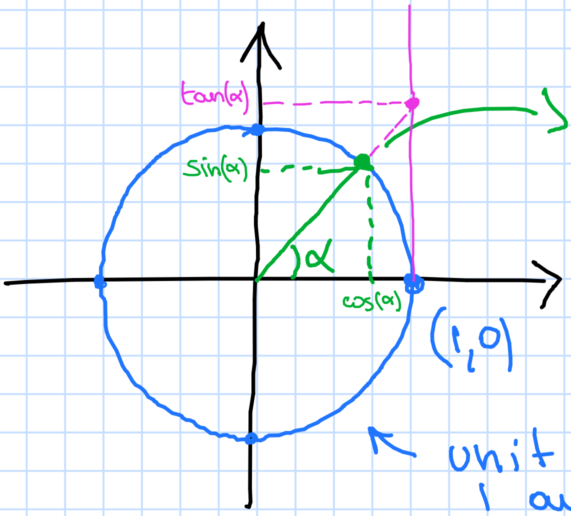
$$\left.\begin{array}{l} f(x) = \sin(x) \\ f(x) = \cos(x) \end{array}\right\} \quad D = \mathbb{R}$$

$$f(x) = \tan(x) \longrightarrow D = \mathbb{R} \setminus \left\{ \dots, -\frac{3}{2}\pi, -\frac{\pi}{2}, \frac{\pi}{2}, \frac{3}{2}\pi, \dots \right\}$$

recall that $\tan(x) = \dfrac{\sin(x)}{\cos(x)}$

hence it is not defined when $\cos(x) = 0$

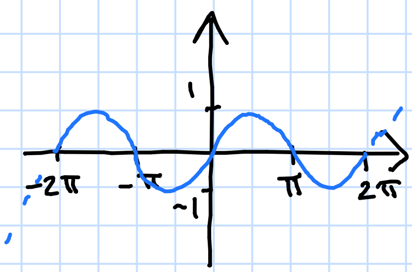
The argument of a trigonometric function is a real number which represents the mesure in radians of the angle



this point has coordinates $(\cos(\alpha), \sin(\alpha))$

unit circle (circle of radius 1 and center (0,0))

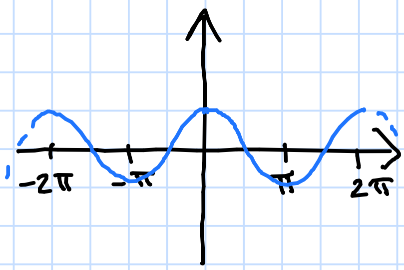
$f(x) = \sin(x)$



odd function

range $= [-1, 1]$

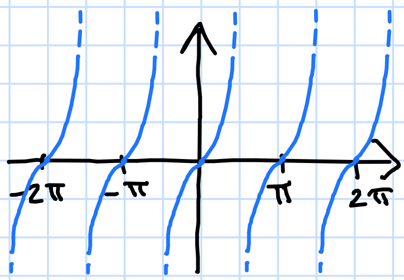
$f(x) = \cos(x)$



even function

range $= [-1, 1]$

$f(x) = \tan(x)$



odd function

range $= \mathbb{R}$

8) exponential function

$f(x) = e^x$

logarithmic function

$f(x) = \ln(x)$

we will study these functions in the second part of the course

Operation with functions

Now that we know the essential functions we can "play" with them to obtain new functions.

Let f, g be two functions with domains respectively D_f and D_g.

- **SUM**: $(f+g)(x) := f(x)+g(x)$ → this means that the value of the sum at some point is equal to the sum of the values of the two functions at that point.

  " is defined as"

  The domain of the new function $f+g$ is the intersection of the two domains.

  $D_{f+g} = D_f \cap D_g$ → indeed both functions has to be defined for computing the value of their sum

- **DIFFERENCE**: $(f-g)(x) := f(x)-g(x)$

  $D_{f-g} = D_f \cap D_g$

- **PRODUCT**: $(f \cdot g)(x) = f(x)g(x)$

  $D_{fg} = D_f \cap D_g$

- **QUOTIENT**: $\frac{f}{g}(x) := \frac{f(x)}{g(x)}$ → indeed if $g(x)=0$ the function $\frac{f}{g}$ is not defined

  $D_{\frac{f}{g}} = \{x \in D_f \cap D_g, \text{ with } g(x) \neq 0\}$

There exists an additional way of combining functions: the operation of composition.

- **COMPOSITION**

  We can compose f and g in two different ways (which are not the same)

$$(f \circ g)(x) := f(g(x))$$

$$(g \circ f)(x) := g(f(x))$$

The best way to understand the composition of function is with an example:

ex: Let $f(x) = x^2$ and $g(x) = x+1$. Then:

$$(f \circ g)(x) = f(g(x)) = (g(x))^2 = (x+1)^2 = x^2+2x+1$$

$$(g \circ f)(x) = g(f(x)) = f(x) + 1 = x+1+1 = x+2$$

Since $f \circ g$ is in general different from $g \circ f$ (as in our case) we have that the operation of composition is not commutative!

ex: $f(x) = \cos(x)$, $g(x) = \sqrt{x}$

$$\Rightarrow \begin{cases} f \circ g(x) = f(g(x)) = \cos(g(x)) = \cos(\sqrt{x}) \\ g \circ f(x) = g(f(x)) = \sqrt{f(x)} = \sqrt{\cos(x)} \end{cases}$$

With all these operations we can built very complicated functions:

ex: $\sqrt{\dfrac{x^2+e^x}{\cos(2x)}} + \tan(|x^3|\sin(x))$

for which we have to make more efforts for finding for example the domain.