Calculus I - MAC 2311 - Section 001

Homework 3 - Solutions

Ex 1. Compute the following limits. If you use l'Hospital's Rule state which type of indeterminate form you have.

a) $\displaystyle\lim_{x\to 1} \frac{\cos(\pi x) + e^{x-1}}{x-1}$

Solution:

$$\lim_{x\to 1} \frac{\cos(\pi x) + e^{x-1}}{x-1} \overset{\frac{0}{0}}{=} \lim_{x\to 1} \frac{\left(\cos(\pi x) + e^{x-1}\right)'}{(x-1)'}$$
$$= \lim_{x\to 1} \frac{-\sin(\pi x)\cdot\pi + e^{x-1}}{1} \overset{\text{plug in}}{=}$$
$$= -\sin(\pi)\cdot\pi + e^{1-1} = 0 + 1 = \mathbf{1}.$$

b) $\displaystyle\lim_{x\to -\infty} \frac{\tan\left(\frac{1}{x}\right) + 1}{\arctan(x)}$

Solution:

We have $\lim_{x\to-\infty} \tan\left(\frac{1}{x}\right) + 1 = \tan(0) + 1 = 1$ and $\lim_{x\to-\infty} \arctan(x) = -\frac{\pi}{2}$. Therefore:

$$\lim_{x\to -\infty} \frac{\tan\left(\frac{1}{x}\right) + 1}{\arctan(x)} = \frac{1}{-\frac{\pi}{2}} = -\frac{\mathbf{2}}{\pi}.$$

c) $\displaystyle\lim_{x\to \infty} (e^x + 1)^{e^{-2x}}$

Solution:

Since $\lim_{x\to\infty} e^x + 1 = \infty$ and $\lim_{x\to\infty} e^{-2x} = 0$, we are faced with the indeterminate form ∞^0. We have:

$$\lim_{x\to\infty} (e^x+1)^{e^{-2x}} \overset{\text{cancellation law}}{=} \lim_{x\to\infty} e^{\ln\left((e^x+1)^{e^{-2x}}\right)} =$$
$$\overset{\text{logarithm law}}{=} \lim_{x\to\infty} e^{e^{-2x}\ln(e^x+1)} =$$
$$\overset{\text{continuity of } e^x}{=} e^{\lim_{x\to\infty} e^{-2x}\ln(e^x+1)}.$$

Now we compute separately $\lim_{x\to\infty} e^{-2x} \ln(e^x+1)$:

$$\lim_{x\to\infty} e^{-2x}\ln(e^x+1) = \lim_{x\to\infty} \frac{\ln(e^x+1)}{e^{2x}} \overset{\frac{\infty}{\infty}}{=} \lim_{x\to\infty} \frac{(\ln(e^x+1))'}{(e^{2x})'} =$$
$$= \lim_{x\to\infty} \frac{\frac{1}{e^x+1}\cdot e^x}{e^{2x}\cdot 2} \overset{\text{simplify}}{=} \lim_{x\to\infty} \frac{\frac{1}{e^x+1}}{e^x\cdot 2} =$$
$$= \lim_{x\to\infty} \frac{1}{(e^x+1)\cdot e^x\cdot 2} = \text{“}\frac{1}{\infty}\text{”} = 0.$$

Therefore we have:

$$\lim_{x\to\infty} (e^x+1)^{e^{-2x}} = e^{\lim_{x\to\infty} e^{-2x}\ln(e^x+1)} = e^0 = \mathbf{1}.$$

d) $\lim_{x\to\infty} \dfrac{\pi - 2\arctan(x)}{e^{-x}}$

Solution:

We have $\lim_{x\to\infty} \pi - 2\arctan(x) = \pi - 2\cdot\frac{\pi}{2} = 0$ and $\lim_{x\to\infty} e^{-x} = 0$, so we are faced with the indeterminate form $\frac{0}{0}$. We have:

$$\begin{aligned}\lim_{x\to\infty} \frac{\pi - 2\arctan(x)}{e^{-x}} &\overset{\frac{0}{0}}{=} \lim_{x\to\infty} \frac{(\pi - 2\arctan(x))'}{(e^{-x})'} = \\ &= \lim_{x\to\infty} \frac{\frac{-2}{1+x^2}}{-e^{-x}} = \lim_{x\to\infty} \frac{2e^x}{1+x^2} \overset{\frac{\infty}{\infty}}{=} \\ &= \lim_{x\to\infty} \frac{2e^x}{2x} \overset{\frac{\infty}{\infty}}{=} \lim_{x\to\infty} \frac{2e^x}{2} = \infty.\end{aligned}$$

e) $\lim_{x\to 0^+} x\cdot\ln(\ln(x+1))$

Solution:

We have $\lim_{x\to 0^+} x = 0$ and $\lim_{x\to 0^+} \ln(\ln(x+1)) =$ "$\ln(0^+)$" $= -\infty$, so we are faced with the indeterminate form $0\cdot\infty$. We have:

$$\begin{aligned}\lim_{x\to 0^+} x\cdot\ln(\ln(x+1)) &= \lim_{x\to 0^+} \frac{\ln(\ln(x+1))}{\frac{1}{x}} = \lim_{x\to 0^+} \frac{(\ln(\ln(x+1)))'}{\left(\frac{1}{x}\right)'} \overset{\frac{\infty}{\infty}}{=} \\ &= \lim_{x\to 0^+} \frac{\frac{1}{\ln(x+1)}\cdot\frac{1}{x+1}}{-\frac{1}{x^2}} = \lim_{x\to 0^+} \frac{-x^2}{(x+1)\cdot\ln(x+1)} \overset{\frac{0}{0}}{=} \\ &= \lim_{x\to 0^+} \frac{(-x^2)'}{((x+1)\cdot\ln(x+1))'} = \lim_{x\to 0^+} \frac{-2x}{\ln(x+1)+(x+1)\cdot\frac{1}{x+1}} = \\ &= \lim_{x\to 0^+} \frac{-2x}{\ln(x+1)+1} = \frac{0}{0+1} = \mathbf{0}.\end{aligned}$$

Ex 2. (25 points) Consider the function

$$f(x) = \frac{1}{x}\cdot e^x.$$

a) Find the domain of definition of f.

b) Find the horizontal and vertical asymptotes.

c) Find the critical numbers of f.

d) Find the intervals over which f is increasing/decreasing and the local maximum/minimum value of f.

e) After having shown that

$$f''(x) = \frac{e^x(x^2-2x+2)}{x^3},$$

find the intervals where f is concave upward/downward and the inflection points of f, if any. (*Hint: note that the equation $x^2 - 2x + 2 = 0$ has no real solutions.*)

f) Sketch the graph of $y = f(x)$, by using the information you collected above.

Solution:

a) *Find the domain of definition of f.*

$D = \mathbb{R}\backslash\{0\} = (-\infty, 0) \cup (0, \infty)$.

b) *Find the horizontal and vertical asymptotes.*

For finding the horizontal and vertical asymptotes we have to study the behavior of the function at the endpoints of the domain, which are in this case $-\infty, 0^-, 0^+, \infty$.

- *Horizontal asymptotes*

  $\lim\limits_{x\to\infty} \frac{1}{x} \cdot e^x = \lim\limits_{x\to\infty} \frac{e^x}{x} \overset{\frac{\infty}{\infty}}{=} \lim\limits_{x\to\infty} \frac{(e^x)'}{(x)'} = \lim\limits_{x\to\infty} \frac{e^x}{1} = \infty.$

  $\lim\limits_{x\to-\infty} \frac{1}{x} \cdot e^x = 0 \cdot 0 = 0.$

  Hence $y = 0$ is the only horizontal asymptote.

- *Vertical asymptotes*

  $\lim\limits_{x\to 0^-} \frac{1}{x} \cdot e^x = \text{“} -\infty \cdot 1\text{”} = -\infty.$

  $\lim\limits_{x\to 0^+} \frac{1}{x} \cdot e^x = \text{“}\infty \cdot 1\text{”} = \infty.$

  Hence 0 is an infinite discontinuity and $x = 0$ is the corresponding vertical asymptote.

c) *Find the critical numbers of f.*

The critical numbers of f are the numbers c in the domain of f where $f'(c) = 0$ or $f'(c)$ does not exist. Let us compute $f'(x)$.

$$f'(x) = \left(\frac{1}{x} \cdot e^x\right)' = -\frac{1}{x^2} \cdot e^x + \frac{1}{x}e^x = e^x \cdot \left(-\frac{1}{x^2} + \frac{1}{x}\right) = e^x \cdot \left(\frac{-1+x}{x^2}\right) = \frac{e^x(x-1)}{x^2}.$$

- $f'(c) = 0$:

  We have that $f'(x) = 0 \Leftrightarrow e^x\left(\frac{-1+x}{x^2}\right) = 0 \overset{e^x \neq 0}{\Leftrightarrow} \frac{-1+x}{x^2} = 0 \Leftrightarrow -1 + x = 0 \Leftrightarrow x = 1$ which is in the domain $D = (-\infty, 0) \cup (0, \infty)$.

- $f'(c)$ does not exist:

  The derivative f' is not defined at $x = 0$, but this point is not in the domain $D = (-\infty, 0) \cup (0, \infty)$.

Hence f has only one critical number: $x = 1$.

d) *Find the intervals over which f is increasing/decreasing and the local maximum/minimum value of f.*

We have to study the sign of the first derivative $f'(x)$. Indeed the function is increasing on the intervals where $f'(x) > 0$ and decreasing on the intervals where $f'(x) < 0$.

	$(-\infty, 0)$	0	$(0,1)$	1	$(1,\infty)$
$f'(x)$	$-$		$-$		$+$
$f(x)$	$\searrow$		$\searrow$		$\nearrow$

Remark: On the real line we mark all the values that make the numerator or the denominator of f' equal to 0. In this case the numerator is $e^x(x-1)$ and the denominator x^2, so that we consider 0 and 1. Now, in order to determine the sign of $f'(x)$ on the intervals $(-\infty, 0)$, $(0, 1)$, $(1, -\infty)$, we simply plug in into $f'(x)$ a number inside the previous intervals and we keep the sign of the obtained value. For example $-1 \in (-\infty, 0)$ and $f'(-1) = \frac{e^{-1}(-2)}{1} < 0$, so that $f'(x) < 0$ on $(-\infty, 0)$.

We conclude that the function f is increasing on the interval $(1, \infty)$ and decreasing on $(-\infty, 0) \cup (0, 1)$. We obtain also that at $x = 1$ the function f has a local minimum value $f(1) = e$.

e) *Find the intervals where f concaves upward/downward and the inflection points of f.*

We have to study the sign of the second derivative $f''(x)$. Indeed the function is concave upward in the intervals where $f''(x) > 0$ and concave downward in the intervals where $f''(x) < 0$.
The inflection points are the points where f is continuous and the graph of f switches from being concave upward to concave downward, or vice versa.
Let us first compute $f''(x)$:

$$
\begin{aligned}
f''(x) &= \left(\frac{e^x(x-1)}{x^2}\right)' = \frac{(e^x(x-1)+e^x)\,x^2 - e^x(x-1)2x}{x^4} = \\
&= \frac{e^x(x^3-x^2)+e^x x^2 - e^x(2x^2-2x)}{x^4} = \frac{e^x(x^3-x^2+x^2-2x^2+2x)}{x^4} = \\
&= \frac{e^x(x^3-2x^2+2x)}{x^4} = \frac{e^x x(x^2-2x+2)}{x^4} = \frac{e^x(x^2-2x+2)}{x^3}
\end{aligned}
$$

	$(-\infty, 0)$	0	$(0,\infty)$
$f''(x)$	$-$		$+$
$f(x)$	DOWN		UP

We conclude that the function f is concave downward on the interval $(-\infty, 0)$ and upward on $(0, \infty)$.

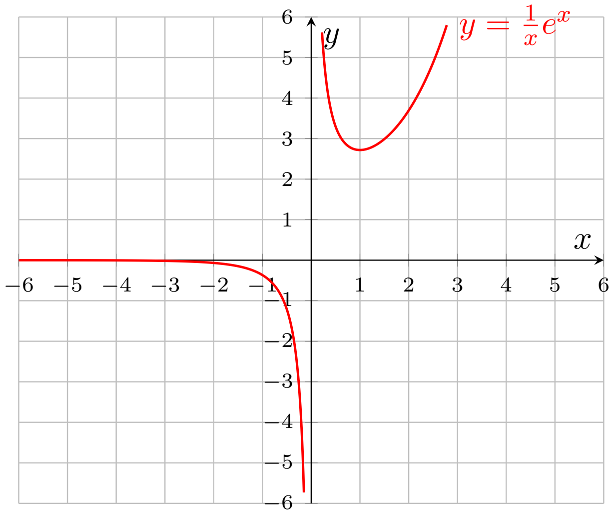
Attention: Even if at $x = 0$ the graph of the function switches from being concave downward to concave upward, this does not correspond to an inflection point, since f is not continuous at $x = 0$ (actually 0 does not belong to the domain of f).

f) *Sketch the graph of $y = f(x)$, by using the information you collected above.*

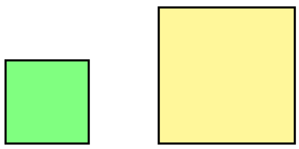
In the previous steps we obtained the following information:

$\star$ $D = \mathbb{R}\backslash\{0\} = (-\infty, 0) \cup (0, \infty)$.

- ⋆ The line $y = 0$ is a horizontal asymptote and $\lim_{x\to-\infty} f(x) = 0$ and $\lim_{x\to\infty} f(x) = \infty$.
- ⋆ The line $x = 0$ is a vertical asymptote and $\lim_{x\to 0^-} f(x) = -\infty$ and $\lim_{x\to 0^+} f(x) = \infty$.
- ⋆ The function f is increasing on the interval $(1, \infty)$ and decreasing on $(-\infty, 0) \cup (0, 1)$. Moreover f has a local minimum value $f(1) = e$ at $x = 1$. Then the graph of f passes through the points $(1, e)$.
- ⋆ The function f is concave downward on the interval $(-\infty, 0)$ and upward on $(0, \infty)$ and there are no inflection points.



Ex 3. (20 points) A farmer has 400 feet of fencing and wants to fence two square fields, each one on all four sides (see the picture below). What are the length of the sides of the two square fields when they cover (together) the least area?



Solution:

Let us call:

x: the side length of the green square (recall that the sides of a square are all of same length...).
y: the side length of the yellow square.

Since the farmer wants to fence each side of the two squares, the total amount of fencing has to be equal to the sum of the perimeters of the two squares. This means that the variables x and y satisfy the following **constraint equation:**

$$4x + 4y = 400 \Leftrightarrow x + y = 100.$$

The farmer wants that the two square fields cover together the least area. Therefore, we want to minimize the sum of the areas of the two squares:

$$\textbf{Total surface area} : x^2 + y^2.$$

Now, all we have to do is obtaining from this function in two variables a function in only one variable (indifferently in x or y), and applying the classical tools for finding the local minimum value(s).

From the constraint equation we get

$$y = 100 - x.$$

If we replace y in the total surface area function $x^2 + y^2$ we obtain the following function in one variable:

$$f(x) = x^2 + (100 - x)^2 = x^2 + 10000 - 200x + x^2 = 2x^2 - 200x + 10000.$$

We want to find at what number(s) $f(x)$ has an absolute minimum value. Thus, let us find the critical points of $f(x)$:

$$f'(x) = 4x - 200 = 0 \Leftrightarrow x = 50.$$

Moreover we have $f''(x) = 4 > 0$, so $f''(50) = 4 > 0$. Then, by the **second derivative test** the function has a local minimum value at $x = 50$.

Hence we obtain that the length of the sides of the two square fields when they cover (together) the least area are $x = 50$ feet and $y =$100 $- 50 =$ 50 feet.

Ex 4. (20 points) A calculus student wakes up late for his calculus test and starts driving to school, trying to make it on time. His home is 10 miles away from USF and the speed limit on all the roads on his way is 35 miles per hour.

a) A webcam located 2 miles away from his home records the car of the student at 9:18 am, and another one located 6 miles away from his home records it at 9:24 am. Prove that the student will be fined for speeding.

b) After 9:24 am the student complies with the speed limit. Show that he will not arrive on time for the test at 9:30 am ☹.

Hint: You can consider the position function $f(t)$, where $f(t)$ represents at a time t the distance of the car of the student from his house. Be careful with the units of measure and recall that 1 min $= \frac{1}{60}$ hour.

Conclusion: Respect always the speed limits and set more than one alarm for the day of your test! ☺.

Solution:

This exercise is an application of the Mean Value Theorem to the everyday life!

First of all note that the position function of an object is always differentiable (at each time the object has an instantaneous velocity).

Let $f(t)$ be the function that represents at a time t the distance of the car of the student from his house.

a) We have $f(9{:}18) = 2$ miles and $f(9{:}24) = 6$ miles. If we apply the Mean Value Theorem to the interval of time between 9:18am and 9:24am we get that there exists a time c between 9:18am and 9:24am such that

$$f'(c) = \frac{f(9{:}24) - f(9{:}18)}{9{:}24 - 9{:}18} = \frac{6-2}{6} = \frac{4\,\text{miles}}{6\,\text{minutes}} = \frac{4\,\text{miles}}{\frac{1}{10}\,\text{hour}} = 40\,\frac{\text{miles}}{\text{hour}}.$$

This means that there is a time c between 9:18am and 9:24am at which the car of the student runs at 40 miles per hour, which is above the speed limit.

b) Between 9:24am and 9:30am the velocity of the student is below 35 miles per hour. If we apply the Mean Value Theorem to the interval of time between 9:24am and 9:30am we get that there exists a time c between 9:24am and 9:30am such that

$$f'(c) = \frac{f(9{:}30) - f(9{:}24)}{9{:}30 - 9{:}24} = \frac{f(9{:}30) - 6\,\text{miles}}{6\,\text{min}} = \frac{f(9{:}30) - 6\,\text{miles}}{\frac{1}{10}\,\text{hour}} \leq 35\frac{\text{miles}}{\text{hour}}.$$

Then

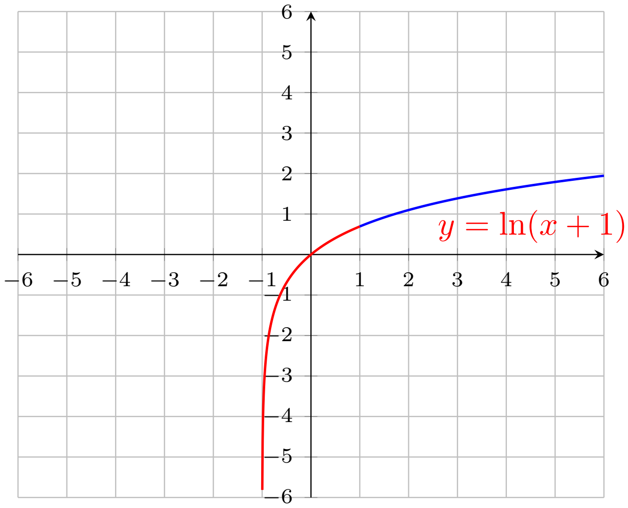
$$f(9{:}30) \leq 35 \cdot \frac{1}{10} + 6 = 3.5 + 6 = 9.5\text{miles}.$$

Hence, at 9:30am the students is at most 9.5 miles away from his house and does not reach the university on time (the university is indeed 10 miles away from his house).

Ex 5. (20 points) Which statements are True/False? Justify your answers.

a) The function $f(x) = \ln(x+1)$ has an absolute maximum and minimum value on $[-1, 1]$.

False. The graph of $f(x)$ is the graph of $\ln(x)$ shifted by 1 towards the left:



Hence the range of $f(x)$ over $[-1, 1]$ is $(-\infty, f(1)] = (-\infty, \ln(2)]$. This means that on $[-1, 1]$ the function $f(x)$ has an absolute maximum value $\ln(2)$ but it has not an absolute minimum value.
Note that the Extreme Value Theorem can not be applied in this case, since the function $\ln(x+1)$ is not continuous on $[-1, 1]$ (it is actually not defined at -1).

b) If f is a function such that $f''(x) > 0$ for all x, and $f'(1) = 1$, then the absolute maximum value of f on the interval $[1, 3]$ is $f(3)$.

True.
Since $f''(x) > 0$ for all x, then $f'(x)$ is an increasing function on $(-\infty, \infty)$. Since $f'(1) = 1$ and f' is increasing, this implies that $f'(x) > 1$ on $(1, \infty)$; in particular $f'(x) > 0$ on $(1, \infty)$. Thus f is increasing on $(1, \infty)$ and the absolute maximum value of f over the closed interval $[1, 3]$ is attained at the right endpoint, that is $f(3)$ is the absolute maximum value of f over $[1, 3]$.

c) The function

$$f(x) = \begin{cases} -x^2, & \text{when } x < 0 \\ x^2 + 1, & \text{when } x \geq 0 \end{cases}$$

has an inflection point at $(0, 1)$ since $f''(x) < 0$ on $(-\infty, 0)$ and $f''(x) > 0$ on $(0, \infty)$.

False.
Even if at $x = 0$ the graph of the function switches from being concave downward ($f''(x) < 0$ before 0) to concave upward ($f''(x) > 0$ after 0), this does not correspond to an inflection point, since f is not continuous at $x = 0$. Indeed we have:

$$\lim_{x \to 0^-} f(x) = \lim_{x \to 0^-} -x^2 = 0 \qquad \lim_{x \to 0^+} f(x) = \lim_{x \to 0^+} x^2 + 1 = 1,$$

so the left-hand limit and the right-hand limit at 0 are not the same.

d) Let f be a function such that $f'(x) \neq 0$ for all x. Then the equation $f(x) = 0$ can have two different solutions x_1 and x_2.

False.
If x_1 and x_2 are two different solutions of the equation $f(x) = 0$, i.e. $f(x_1) = 0$ and $f(x_2) = 0$ with $x_1 \neq x_2$, then by the Mean Value Theorem applied to the interval $[x_1, x_2]$ (or Rolle's theorem) there exists c in (x_1, x_2) such that

$$f'(c) = \frac{f(x_2) - f(x_1)}{x_2 - x_1} = \frac{0}{x_2 - x_1} = 0,$$

but this is not possible, since $f'(x) \neq 0$ for all x.