Calculus I - MAC 2311 - Section 001

Homework 2 - Solutions

Ex 1. (24 points) Differentiate with respect to the indicated variable. If k appears in the function, treat it as a constant. Before starting computing your derivative, think if it is possible to simplify the function. Show all your work.

a) $\frac{d}{dx}\left[2x^7 - 3x^5 - 5x^3 + 7x^2\right] = 14x - 15x^4 - 15x^2 + 14x.$

b) $\frac{d}{du}\left[-\frac{u^3}{u^{11}} + \frac{\sqrt[3]{u^2}}{\sqrt[5]{u^6}}\right] = \frac{d}{du}\left[-u^{3-11} + \frac{u^{\frac{2}{3}}}{u^{\frac{6}{5}}}\right] = \frac{d}{du}\left[-u^{-8} + u^{\frac{2}{3}-\frac{6}{5}}\right] = \frac{d}{du}\left[-u^{-8} + u^{-\frac{8}{15}}\right] =$
$-(-8u^{-9}) + \left(-\frac{8}{15}u^{\frac{8}{15}-1}\right) = 8u^{-9} - \frac{8}{15}u^{-\frac{7}{15}} = \frac{8}{u^9} - \frac{8}{15\sqrt[15]{u^7}}.$

c) $\frac{d}{dx}\left[\frac{x^3 + 5x\sin(x)}{x}\right] = \frac{d}{dx}\left[\frac{x\left(x^2 + 5\sin(x)\right)}{x}\right] \overset{x\neq 0}{=} \frac{d}{dx}\left[x^2 + 5\sin(x)\right] = 2x - 5\cos(x).$

Hence $2x - 5\cos(x)$ is the derivative of $\frac{x^3+5x\sin(x)}{x}$ for all $x \neq 0$ (note that since the function $\frac{x^3+5x\sin(x)}{x}$ is not defined at 0, it is not differentiable at 0).

d) $\frac{d}{dt}\left[t^2\cos(t)\right] \overset{\text{product rule}}{=} \frac{d}{dt}(t^2)\cdot\cos(t) + t^2\frac{d}{dt}(\cos(t)) = 2t\cos(t) - t^2\sin(t).$

e) $\frac{d}{dx}\left[\frac{e^{\sqrt{2018}}}{\pi^3}\right] = 0$, since $\frac{e^{\sqrt{2018}}}{\pi^3}$ is a constant.

f) $\frac{d}{dx}\left[\frac{x^2}{\ln(x)}\right] \overset{\text{quotient rule}}{=} \frac{\frac{d}{dx}(x^2)\cdot\ln(x) - x^2\cdot\frac{d}{dx}(\ln(x))}{\ln^2(x)} = \frac{2x\ln(x) - x^2\cdot\frac{1}{x}}{\ln^2(x)} = \frac{2x\ln(x) - x}{\ln^2(x)}.$

g) $\frac{d}{d\theta}\left[\sqrt{\cos(\theta)+1}\right] = \frac{d}{d\theta}\left[(\cos(\theta)+1)^{\frac{1}{2}}\right] \overset{\text{chain rule}}{=} \frac{1}{2}(\cos(\theta)+1)^{-\frac{1}{2}}\frac{d}{d\theta}(\cos(\theta)+1) =$
$= \frac{1}{2}(\cos(\theta)+1)^{-\frac{1}{2}}(-\sin(\theta)) = -\frac{1}{2}(\cos(\theta)+1)^{-\frac{1}{2}}\sin(\theta).$

h) $\frac{d}{d\theta}\left[k\cos\left(\sqrt{\theta}\right)+1\right] = k\frac{d}{d\theta}\left(\cos\left(\sqrt{\theta}\right)\right) + \frac{d}{d\theta}(1) \overset{\text{chain rule}}{=} k\left(-\sin(\sqrt{\theta}\right)\frac{d}{d\theta}(\sqrt{\theta}) + 0 =$
$= -k\sin(\sqrt{\theta})\cdot\frac{1}{2}\theta^{-\frac{1}{2}} = -\frac{k}{2}\sin(\sqrt{\theta})\cdot\theta^{-\frac{1}{2}}.$

i) $\frac{d}{dx}\left[e^{2\cos(x)}\right] \overset{\text{chain rule}}{=} e^{2\cos(x)}\frac{d}{dx}(2\cos(x)) = e^{2\cos(x)}\cdot(-2\sin(x)) = -2e^{2\cos(x)}\sin(x).$

j) $\frac{d}{d\alpha}\left[\tan(\sin(\pi\alpha))\right] \overset{\text{chain rule 1}}{=} \sec^2(\sin(\pi\alpha))\cdot\frac{d}{d\alpha}(\sin(\pi\alpha)) \overset{\text{chain rule 2}}{=}$
$= \sec^2(\sin(\pi\alpha))\cdot\cos(\pi\alpha)\cdot\frac{d}{d\alpha}(\pi\alpha) = \sec^2(\sin(\pi\alpha))\cdot\cos(\pi\alpha)\cdot\pi = \pi\sec^2(\sin(\pi\alpha))\cdot\cos(\pi\alpha).$

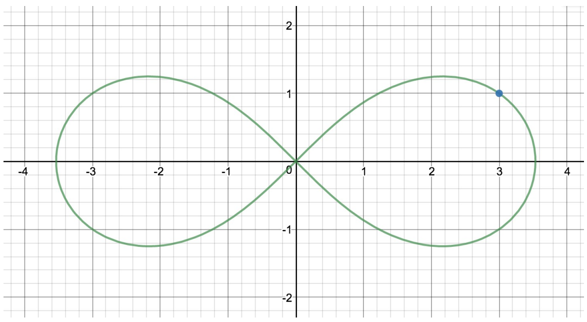
k) $\frac{d}{dx}\left[e^{\ln(\tan(x))}\right] \overset{e^{\ln(x)}=x}{=} \frac{d}{dx}\left[\tan(x)\right] = \sec^2(x).$

Note that the cancellation equation $e^{\ln(x)} = x$ is true for all $x > 0$. Thus, if we want to be precise, we should point out that the simplification $e^{\ln(\tan(x))} = \tan(x)$ is true when $\tan(x) > 0$. As a consequence $\sec^2(x)$ is the derivative of $e^{\ln(\tan(x))}$ for all x such that $\tan(x) > 0$ (note again that since the function $e^{\ln(\tan(x))}$ is not defined when $\tan(x) \leq 0$, it is not differentiable at those points).

l) $\frac{d}{dt}\left[\ln\left(e^{\sqrt{k}e^t}\right)\right] \overset{\ln(e^x)=x}{=} \frac{d}{dt}\left[\sqrt{k}e^t\right] = \sqrt{k}\frac{d}{dt}\left(e^t\right) = \sqrt{k}e^t.$
In this case there are no problems since the cancellation equation $\ln(e^x) = x$ is true for all x in $\mathbb{R}$.

Ex 2. (10+10 points) Consider the **lemniscate** curve given by the following equation:

$$2x^4 + 4x^2y^2 + 2y^4 = 25x^2 - 25y^2.$$



a) Use implicit differentiation to find y' (i.e. $\frac{dy}{dx}$).

b) Find an equation of the tangent line to the above curve at the point $(3, 1)$.

Solution:

a) We take the derivative of each side of the equation of the curve with respect to x (recall to treat y as a function of x), and apply the rules of differentiation:

$$\frac{d}{dx}\left(2x^4 + 4x^2y^2 + 2y^4\right) = \frac{d}{dx}\left(25x^2 - 25y^2\right)$$

$$\Downarrow \text{ sum rule}$$

$$\frac{d}{dx}(2x^4) + \frac{d}{dx}(4x^2y^2) + \frac{d}{dx}(2y^4) = \frac{d}{dx}(25x^2) - \frac{d}{dx}(25y^2)$$

$$\Downarrow \text{ product rule+chain rule}$$

$$8x^3 + \left[\frac{d}{dx}(4x^2) \cdot y^2 + 4x^2 \cdot \frac{d}{dx}(y^2)\right] + 8y^3 \cdot \frac{dy}{dx} = 50x - 50y \cdot \frac{dy}{dx}$$

$$\Downarrow$$

$$8x^3 + 8xy^2 + 4x^2 \cdot 2y \cdot \frac{dy}{dx} + 8y^3 \cdot \frac{dy}{dx} = 50x - 50y \cdot \frac{dy}{dx}$$

Now we have an ordinary linear equation where the unknown we want to solve for is $\frac{dy}{dx}$. From the last step we obtain:

$$8x^2y \cdot \frac{dy}{dx} + 8y^3 \cdot \frac{dy}{dx} + 50y \cdot \frac{dy}{dx} = 50x - 8x^3 - 8xy^2$$
$$\Downarrow$$
$$(8x^2y + 8y^3 + 50y) \cdot \frac{dy}{dx} = 50x - 8x^3 - 8xy^2$$

which implies

$$\frac{dy}{dx} = \frac{50x - 8x^3 - 8xy^2}{8x^2y + 8y^3 + 50y}.$$

b) If $P(x, y)$ is a point on the lemniscate we have that the slope of the tangent line to the curve at $P(x, y)$ is given by:

$$\frac{dy}{dx} = \frac{50x - 8x^3 - 8xy^2}{8x^2y + 8y^3 + 50y}.$$

Hence, for the point $(3, 1)$, by substituting $x = 3$ and $y = 1$ in the previous formula, we get:

$$\frac{dy}{dx} = \frac{50 \cdot 3 - 8 \cdot 3^3 - 8 \cdot 3 \cdot 1^2}{8 \cdot 3^2 \cdot 1 + 8 \cdot 1^3 + 50 \cdot 1} = \frac{150 - 216 - 24}{72 + 8 + 50} = \frac{-90}{130} = -\frac{9}{13}.$$

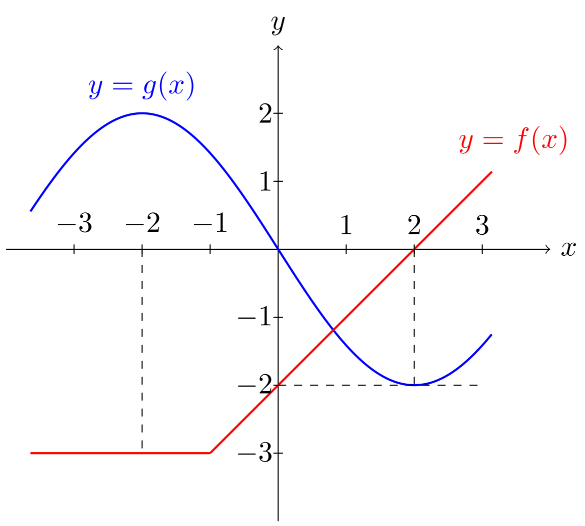
We deduce that an equation of the tangent line to the lemniscate at the point $(3, 1)$ is

$$y - 1 = -\frac{9}{13} \cdot (x - 3),$$

i.e.

$$y = -\frac{9}{13}x + \frac{40}{13}.$$

Ex 3. (5+5+5+5 points)



Let f and g be the functions whose graphs are shown above and let

$$h(x) = f(x) + g(x), \qquad u(x) = f(x)g(x), \qquad v(x) = \frac{f(x)}{g(x)}, \qquad w(x) = f(g(x)).$$

Compute $h'(2)$, $u'(2)$, $v'(2)$ and $w'(2)$, without finding explicit formula for $f(x)$ and $g(x)$.

Solution:

By using the differentiation rules (respectively sum, product, quotient and chain rule) we have:

$$\begin{aligned} h'(x) &= f'(x) + g'(x); \\ u'(x) &= f'(x)g(x) + f(x)g'(x); \\ v'(x) &= \frac{f'(x)g(x) - f(x)g'(x)}{(g(x))^2}; \\ w'(x) &= f'(g(x))g'(x). \end{aligned}$$

Hence, in order to compute $h'(2)$, $u'(2)$, $v'(2)$ and $w'(2)$, we need before to find the values for $f(2)$, $g(2)$, $f'(2)$, $g'(2)$, $f'(g(2))$.

- Easily from the graphs of f and g we get that $f(2) = 0$ and $g(2) = -2$.
- For computing $f'(2)$ (respectively $g'(2)$) we need to find the slope of the tangent line to the graph $y = f(x)$ (respectively $y = g(x)$) at the point $(2, f(2))$ (respectively $(2, g(2))$).

  In the first case, the graph $y = f(x)$ is a line, which is tangent to itself at each point. Thus, we can compute its slope by using the coordinates of two of its points, for example $(2, 0)$ and $(0, -2)$, and we have:

  $$f'(2) = \frac{-2 - 0}{0 - 2} = 1.$$

  In the second case, the tangent line to $y = g(x)$ at $(2, g(2))$ is horizontal (parallel to the x-axis), so that its slope is 0. This means that

  $$g'(2) = 0.$$

- Finally we have:

  $$f'(g(2)) = f'(-2) = 0$$

  since the tangent line to $y = f(x)$ at $(-2, f(-2))$ is again horizontal.

We are now ready for computing $h'(2)$, $u'(2)$, $v'(2)$ and $w'(2)$:

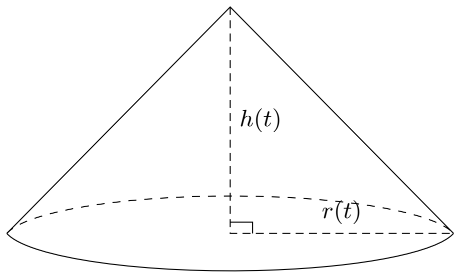
$$\begin{aligned} h'(2) &= f'(2) + g'(2) = 1 + 0 = 1; \\ u'(2) &= f'(2)g(2) + f(2)g'(2) = 1 \cdot (-2) + 0 \cdot 0 = -2; \\ v'(2) &= \frac{f'(2)g(2) - f(2)g'(2)}{(g(2))^2} = \frac{1 \cdot (-2) - 0 \cdot 0}{(-2)^2} = \frac{-2}{4} = -\frac{1}{2}; \\ w'(2) &= f'(g(2))g'(2) = 0 \cdot 0 = 0. \end{aligned}$$

Ex 4. (20 points) It is the Sunday before the second test. The calculus student of HW1, who was disappointed by his experience on Floridian mountains, decides this time to have a productive study break at Clearwater beach. After filling a bucket with dry sand, he starts pouring the sand on the ground at a steady rate of 5 cm^3/s. He notices that, at each time, the sand forms a conical pile whose height is always equal to half of the diameter of its base. How fast is the radius of the conical pile increasing when the height is 10 cm?

(You can find a suitable formula for the volume in the "Geometry" section of Reference Page 1 at the end of your textbook.)

Solution:

♦ **Step 1:** *Understand the problem - Draw a picture - Find and name the quantities which are related.*



At a given time t let:

- $V(t)$: the volume of the cone of sand (equal to the volume of sand that has fallen from the bucket);
- $r(t)$: the radius of the cone of sand;
- $h(t)$: the height of the cone of sand.

♦ **Step 2:** *Write what you know and what you wish to find!*

- **Known:** $\frac{dV}{dt} = 5\,\frac{\text{cm}^3}{\text{sec}}$ and $h(t) = r(t)$ fot all t.
- **Unknown:** $\frac{dr}{dt}$ when $h(t) = 10$ cm.

♦ **Step 3:** *Find how the quantities are related (i.e. find a suitable equation which relates the quantities).*

At each time t the quantities $V(t)$, $h(t)$ and $r(t)$ are related by the equation given by the formula of the volume of a cone:

$$V(t) = \frac{\pi}{3} r^2(t) h(t)$$
$$\Downarrow \; r(t) = h(t)$$
$$V(t) = \frac{\pi}{3} r^3(t)$$

♦ **Step 4:** *Differentiate the above equation (so that the related quantities will give you the related rates).*

$$\frac{d}{dt}(V(t)) = \frac{d}{dt}\left(\frac{\pi}{3}r^3(t)\right)$$
$$\Downarrow$$
$$\frac{dV}{dt} = \frac{\pi}{3}3r^2(t)\cdot\frac{dr}{dt}$$
$$\Downarrow$$
$$\frac{dV}{dt} = \pi r^2(t)\cdot\frac{dr}{dt}$$

♦ **Step 5:** *Solve for the unknown quantity and replace the known data (with unit of measures).*

From the last equation we get:

$$\frac{dr}{dt} = \frac{1}{\pi r^2(t)}\frac{dV}{dt}.$$

By replacing in the previous equation all the known data (note that when $h(t) = 10$ cm then also $r(t) = 10$ cm, since radius and height of the cone are the same at each time) we get:

$$\frac{dr}{dt} = \frac{1}{\pi r^2(t)}\frac{dV}{dt} = \frac{1}{\pi(10\text{cm})^2}\cdot 5\,\frac{\text{cm}^3}{\text{sec}} = \frac{5}{\pi 100}\,\frac{\text{cm}^3}{\text{cm}^2\text{sec}} = \frac{1}{\pi 20}\,\frac{\text{cm}}{\text{sec}}.$$

Ex 5. (5+5+5+5 points) Which statements are True/False? Justify your answers.

a) If $f(x) = x^{\tan(x)}$ then $f'(x) = \tan(x)\cdot x^{\tan(x)-1}$.

False. The function $f(x) = x^{\tan(x)}$ is not a power function, so we can not apply the power rule. We can compute the derivative in two different ways:

⋆ **I method** : Logarithmic differentiation

$$y = x^{\tan(x)}$$
$$\ln(y) = \ln\left(x^{\tan(x)}\right) = \tan(x)\ln(x)$$
$$\frac{d}{dx}[\ln(y)] = \frac{d}{dx}[\tan(x)\ln(x)]$$
$$\frac{1}{y}\cdot\frac{dy}{dx} = \sec^2(x)\ln(x) + \frac{\tan(x)}{x}$$
$$\frac{dy}{dx} = y\left(\sec^2(x)\ln(x) + \frac{\tan(x)}{x}\right)$$
$$\frac{dy}{dx} = x^{\tan(x)}\left(\sec^2(x)\ln(x) + \frac{\tan(x)}{x}\right).$$

⋆ **II method**

By using the identity $e^{\ln(x)} = x$, we can rewrite the function in the following way:

$$f(x) = x^{\tan(x)} = e^{\ln\left(x^{\tan(x)}\right)} = e^{\tan(x)\ln(x)}.$$

Hence we have:

$$\begin{aligned} f'(x) &= \left(e^{\tan(x)\ln(x)}\right)' = \\ &= e^{\tan(x)\ln(x)}\left(\tan(x)\ln(x)\right)' = \\ &= e^{\tan(x)\ln(x)}\left(\sec^2(x)\ln(x) + \frac{\tan(x)}{x}\right) = \\ &= x^{\tan(x)}\left(\sec^2(x)\ln(x) + \frac{\tan(x)}{x}\right). \end{aligned}$$

b) If $f(x) = \sin(x)$ then $f'''(0) = 0$.

False. We have $f'(x) = \cos(x)$, $f''(x) = (f'(x))' = (\cos(x))' = -\sin(x)$ and $f'''(x) = (f''(x))' = (-\sin(x))' = -\cos(x)$ so that $f'''(0) = -\cos(0) = -1$.

c) If the graph of a physical quantity F as function of time is a line, then the rate of change of F with respect to time is constant.

True. Indeed, at each time t the (instantaneous) rate of change of F with respect to time is nothing else than the slope of the tangent line to the graph of $F(t)$ at the point $(t, F(t))$. Now if the graph of $F(t)$ is a line, which is the tangent line to itself at each point, then the rate of change is given by the slope of this line and it is therefore constant.

d) If $f(x) = e^{x+2}$, then $f^{-1}(e^2) = \{0\}$.

True. Recall that $f^{-1}(e^2)$ is the set of numbers x such that $f(x) = e^2$. First we note that 0 belongs to $f^{-1}(e^2)$, since $f(0) = e^2$. Moreover, since the function $f(x)$ is one-to one (it is a horizontal shift of the function e^x, which is one-to one), the number 0 can be the only element in $f^{-1}(e^2)$, otherwise the function $f(x)$ would not pass the horizontal line test.