Calculus I - MAC 2311 - Section 003

Quiz 8 - Solutions

11/19/2018

Compute the following limits:

1) $\lim_{x\to 0} \frac{\cos(x) - e^x}{\sin(x) + 2x}$

Solution:

We have $\lim_{x\to 0} \cos(x) - e^x = \cos(0) - e^0 = 1 - 1 = 0$ and $\lim_{x\to 0} = \sin(0) + 2\cdot 0 = 0$, so we are faced with the indeterminate form $\frac{0}{0}$. Hence we can apply L'Hospital's Rule:

$$\lim_{x\to 0} \frac{\cos(x) - e^x}{\sin(x) + 2x} = \lim_{x\to 0} \frac{(\cos(x) - e^x)'}{(\sin(x) + 2x)'} = \lim_{x\to 0} \frac{-\sin(x) - e^x}{\cos(x) + 2} = \frac{-\sin(0) - e^0}{\cos(0) + 2} = -\mathbf{\frac{1}{3}}.$$

2) $\lim_{x\to\infty} x^3 e^{-x^2}$

Solution:

We have $\lim_{x\to\infty} x^3 = \infty$ and $\lim_{x\to\infty} e^{-x^2} = 0$, so that we are faced with the indeterminate form $\infty \cdot 0$. Hence we rewrite the limit in the following way:

$$\lim_{x\to\infty} x^3 e^{-x^2} = \lim_{x\to\infty} \frac{x^3}{e^{x^2}}$$

Now the indeterminate form is $\frac{\infty}{\infty}$ and we can apply L'Hospital's Rule:

$$\lim_{x\to\infty} \frac{x^3}{e^{x^2}} = \lim_{x\to\infty} \frac{\left(x^3\right)'}{\left(e^{x^2}\right)'} = \lim_{x\to\infty} \frac{3x^2}{e^{x^2}\cdot 2x} \overset{\text{simplify}}{=} \lim_{x\to\infty} \frac{3x}{e^{x^2}\cdot 2}.$$

Again, we are faced with the indeterminate form $\frac{\infty}{\infty}$, therefore we apply a second time L'Hospital's Rule:

$$\lim_{x\to\infty} \frac{3x}{2e^{x^2}} = \lim_{x\to\infty} \frac{(3x)'}{\left(2e^{x^2}\right)'} = \lim_{x\to\infty} \frac{3}{2e^{x^2}\cdot 2x} = \text{“}\frac{1}{\infty}\text{”} = \mathbf{0}.$$

3) $\lim_{x\to 0^+} (e^x + x)^{\frac{1}{x}}$

Solution:

We have $\lim_{x\to 0^+} e^x + x = 1$ and $\lim_{x\to 0^+} \frac{1}{x} = \infty$, so that we are faced with the indeterminate form 1^∞. Hence we rewrite the limit in the following way:

$$\lim_{x\to 0^+} (e^x + x)^{\frac{1}{x}} \overset{y=e^{\ln(y)}}{=} \lim_{x\to 0^+} e^{\ln\left((e^x+x)^{\frac{1}{x}}\right)} \overset{\text{logarithm law}}{=} \lim_{x\to 0^+} e^{\frac{1}{x}\ln(e^x+x)} \overset{\text{continuity of } e^x}{=} e^{\lim_{x\to 0^+} \frac{1}{x}\cdot\ln(e^x+x)}.$$

Now we compute separately $\lim_{x\to 0^+} \frac{1}{x} \cdot \ln(e^x + x)$:

$$\lim_{x\to 0^+} \frac{1}{x} \cdot \ln(e^x + x) = \lim_{x\to 0^+} \frac{\ln(e^x + x)}{x} \overset{\frac{0}{0}}{=} \lim_{x\to 0^+} \frac{(\ln(e^x + x))'}{(x)'} = \lim_{x\to 0^+} \frac{\frac{e^x+1}{e^x+x}}{1} = \frac{1+1}{1+0} = 2.$$

Therefore we have:

$$\lim_{x\to 0^+} (e^x + x)^{\frac{1}{x}} = e^{\lim_{x\to 0^+} \frac{1}{x} \ln(e^x+x)} = \mathbf{e^2}.$$

4) A student writes:

$$\lim_{x\to 0^+} \frac{e^x + 1}{x} = \lim_{x\to 0^+} \frac{(e^x + 1)'}{(x)'} = \lim_{x\to 0^+} \frac{e^x}{1} = 1.$$

Do you agree or disagree with the student? Justify your answer. Moreover, if you disagree compute the correct value of the limit.

Solution:

The student can not apply L'Hospital's rule, since the limit is not of the form $\frac{0}{0}$ or $\frac{\infty}{\infty}$. One can instead solve the limit directly and get:

$$\lim_{x\to 0^+} \frac{e^x + 1}{x} = \frac{1}{0^+} = \infty$$