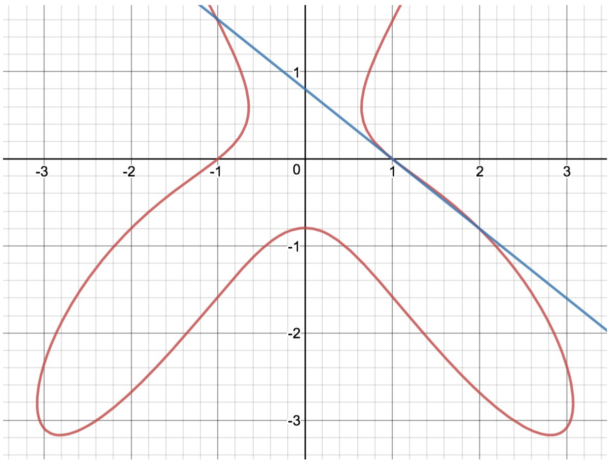
Calculus I - MAC 2311 - Section 003

Quiz 4 - Solutions

10/03/2018

1) Consider the curve $\mathcal{C}$ given by the equation

$$x^4 - 2y^3 = 1 - 5x^2y.$$



a) On the picture above, draw the tangent line to $\mathcal{C}$ at the point $(1, 0)$.

b) Use implicit differentiation to find $\frac{dy}{dx}$.

We take the derivative of each side of the equation of the curve with respect to x (recall to treat y as a function of x), and apply the rules of differentiation:

$$\frac{d}{dx}\left(x^4 - 2y^3\right) = \frac{d}{dx}\left(1 - 5x^2y\right)$$

$\Downarrow$ sum rule

$$\frac{d}{dx}x^4 - \frac{d}{dx}2y^3 = \frac{d}{dx}1 - \frac{d}{dx}(5x^2y)$$

$\Downarrow$ product rule+chain rule

$$4x^3 - 6y^2 \cdot \frac{dy}{dx} = 0 - \left[\frac{d}{dx}(5x^2) \cdot y + 5x^2 \cdot \frac{d}{dx}(y)\right]$$

$\Downarrow$

$$4x^3 - 6y^2 \cdot \frac{dy}{dx} = -10xy - 5x^2 \cdot \frac{dy}{dx}$$

Now we have an ordinary linear equation where the unknown we want to solve for is $\frac{dy}{dx}$. From the last step we obtain:

$$5x^2 \cdot \frac{dy}{dx} - 6y \cdot \frac{dy}{dx} = -4x^3 - 10xy$$
$$\Downarrow$$
$$(5x^2 - 6y) \cdot \frac{dy}{dx} = -4x^3 - 10xy$$

which implies

$$\frac{dy}{dx} = \frac{-4x^3 - 10xy}{5x^2 - 6y}.$$

c) Find an equation of the tangent line to the above curve at the point $(1, 0)$.

If $P(x, y)$ is a point on the curve $\mathcal{C}$, i.e. the coordinates x and y of P make the equation of $\mathcal{C}$ true, we have that the slope of the tangent line to the curve $\mathcal{C}$ at $P(x, y)$ is given by:

$$\frac{dy}{dx} = \frac{-4x^3 - 10xy}{5x^2 - 6y}.$$

Hence, for the point $(1, 0)$, by substituting $x = 1$ and $y = 0$ in the previous formula, we get:

$$\left.\frac{dy}{dx}\right|_{\substack{x=1\\y=0}} = \frac{-4 \cdot 1 - 10 \cdot 1 \cdot 0}{5 \cdot 1 - 6 \cdot 0} = -\frac{4}{5}.$$

We deduce that an equation of the tangent line to the curve $\mathcal{C}$ at the point $(1, 0)$ is

$$y - 0 = -\frac{4}{5} \cdot (x - 1),$$

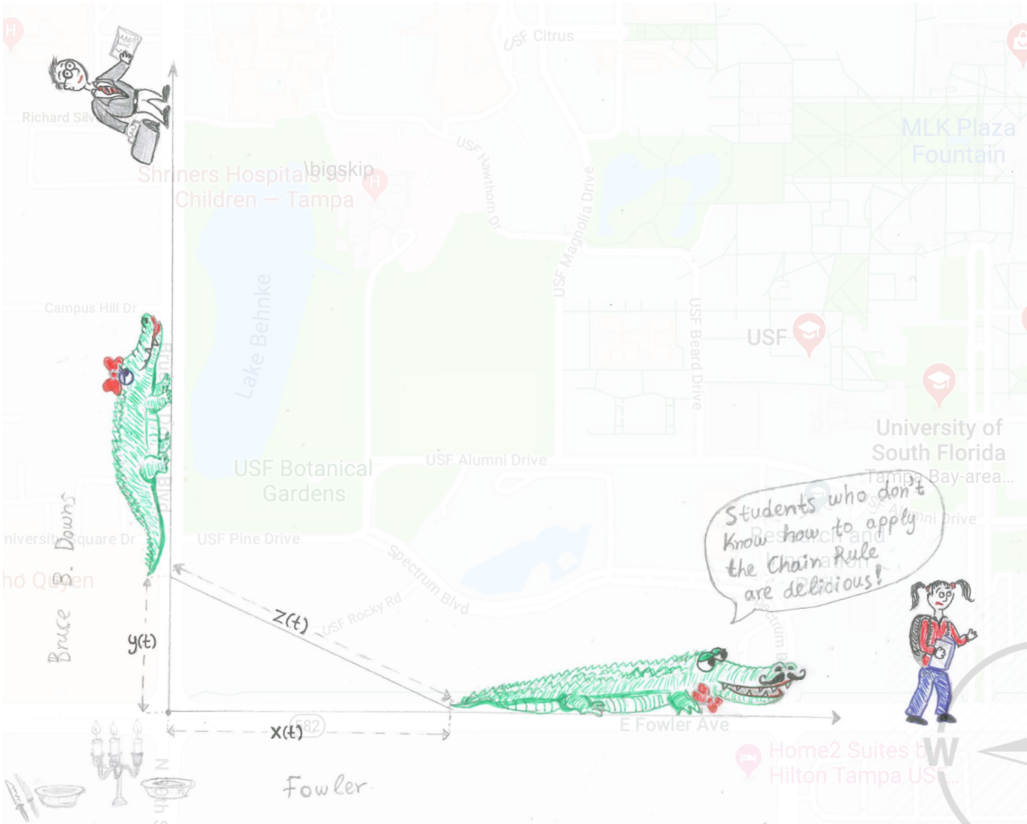
i.e.

$$y = -\frac{4}{5}x + \frac{4}{5}.$$

d) Is your answer for (c) consistent with your "answer" for (a)? Why or why not?

Yes, indeed the line drawn in part (a) has negative slope and approximately equal to $-\frac{4}{5}$.

2) A couple of alligators meets at the intersection of Bruce B. Downs Blvd and Fowler Ave for organizing a romantic dinner. The male alligator starts running east at a speed of 0.4 miles per minute to chase a USF student. At the same time the female alligator starts running north at a speed of 0.3 miles per minute to chase a USF instructor. At what rate is the distance between the two alligators increasing after 5 minutes?



a) Sketch quickly the geometric situation described by the problem on the map above.

b) Name and describe the quantities of the problem (and attach them the corresponding units).

At a given time t:

$x(t)$: the distance between the male alligator and the intersection point;
$y(t)$: the distance between the female alligator and the intersection point;
$z(t)$: the distance between the two alligators.

c) Write what you know and what you want to find.

Known: for all t, $\dfrac{dx}{dt} = 0.4$ miles/min and $\dfrac{dy}{dt} = 0.3$ miles/min

Want to find: $\left.\dfrac{dz}{dt}\right|_{t=5}$.

d) Write an equation that relates the quantities found in (b).

By Pythagoras Theorem the quantities $x(t)$, $y(t)$ and $z(t)$ are related by the following equation:

$$(x(t))^2 + (y(t))^2 = (z(t))^2, \quad \text{for all } t. \tag{1}$$

e) Solve the problem (do not forget the units in your final answer).

Equation (1) shows how the quantities are related at each time t. We are interested in how the corresponding rates relate. For that, we differentiate both sides of equation (1) with respect to t:

$$\frac{d}{dt}(z(t))^2 = \frac{d}{dt}(x(t))^2 + \frac{d}{dt}(y(t))^2 \overset{\text{chain rule}}{\Rightarrow} 2z(t)\cdot\frac{dz}{dt} = 2x(t)\cdot\frac{dx}{dt} + 2y(t)\cdot\frac{dy}{dt}$$

By isolating $\frac{dz}{dt}$ in the last equation we get:

$$\frac{dz}{dt} = \frac{2x(t)\cdot\frac{dx}{dt} + 2y(t)\cdot\frac{dy}{dt}}{2z(t)} = \frac{x(t)\cdot\frac{dx}{dt} + y(t)\cdot\frac{dy}{dt}}{z(t)} \tag{2}$$

which evaluated at $t = 5$ gives:

$$\left.\frac{dz}{dt}\right|_{t=5} = \frac{x(5)\cdot\left.\frac{dx}{dt}\right|_{t=5} + y(5)\cdot\left.\frac{dy}{dt}\right|_{t=5}}{z(5)}$$

Now, for computing $x(5)$, $y(5)$ and $z(5)$, notice that since the alligators are moving at a constant velocity (0.4 miles/minute in the case of the male alligator and 0.3 miles/minutes in the case of the female alligator) we have:

$$x(t) = 0.4t \qquad \text{and} \qquad y(t) = 0.3t$$

Hence

$$x(5) = 0.4\cdot 5 = 2 \text{ miles} \qquad \text{and} \qquad y(5) = 0.3\cdot 5 = 1.5 \text{ miles}.$$

For finding $z(5)$ we use the equation (1) for $t = 5$:

$$z(5) = \sqrt{(x(5))^2 + (y(5))^2} = \sqrt{2^2 + 1.5^2} = \sqrt{4 + 2.25} = \sqrt{6.25} = 2.5 \text{ miles}$$

In conclusion:

$$\left.\frac{dz}{dt}\right|_{t=5} = \frac{x(5)\cdot 0.4 + y(5)\cdot 0.3}{z(5)} = \frac{2\cdot 0.4 + \cdot 1.5\cdot 0.3}{2.5} = 0.5 \text{ miles/minute}.$$

We conclude that after 5 minutes the distance between the two alligators is increasing at a rate of 0.5 miles/minute.