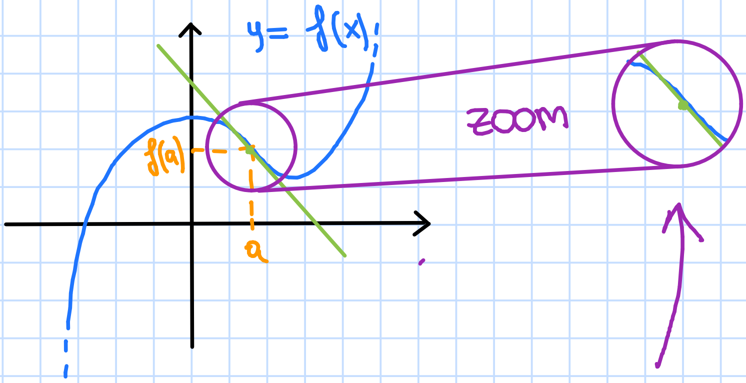
LINEAR APPROXIMATION (Sec. 2.8)

Sometimes it is hard to evaluate the value of a function at a specific point without the use of a calculator. Nevertheless, in some particular cases, we can try to approximate it!

Indeed, imagine that you have a function f which is differentiable at a point a and for which $f(a)$ is easy to compute.
Since f is differentiable at a, we can draw the tangent line to the graph $y = f(x)$ at the point $(a, f(a))$:



We notice that the graph of f lies very close to its tangent line near the point of tangency.

So we can use the equation of the tangent line for approximating the value of f near a: this method is called "linear approximation" of f at a.

The tangent line to the graph $y=f(x)$ at the point $(a, f(a))$ has equation:

$$y - f(a) = f'(a)(x-a)$$

that is

$$y = f(a) + f'(a)(x-a)$$

The function

$$L(x) = f(a) + f'(a)(x-a)$$

note that this is a linear function

is called linearization of f at a.

Now, if x is an input near a then:

$$f(x) \approx L(x)$$

this is called linear approximation of f at a

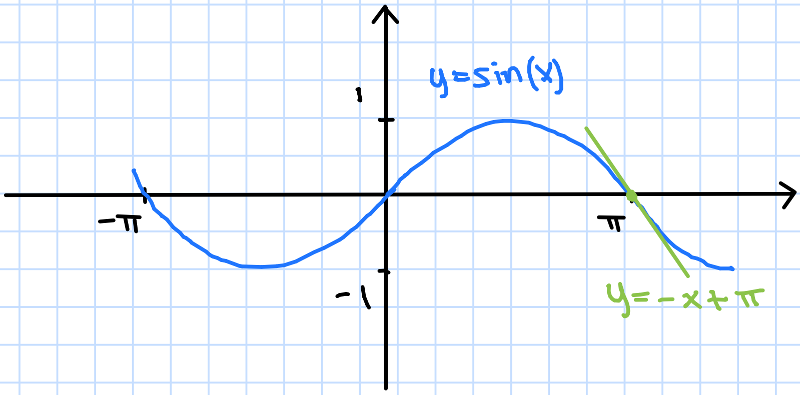
Example 1

1) Find the linearization of $f(x) = \sin x$ at $x = \pi$.
2) Use 1) to approximate the number $\sin(3)$.

Solution

1) The tangent line to the graph of $f(x)$ at $x=\pi$ is given by:

$$y - f(\pi) = f'(\pi)\,(x-\pi)$$
$$y - \sin(\pi) = \cos(\pi)\,(x-\pi)$$
$$y - 0 = -1\,(x-\pi)$$
$$y = -x + \pi$$



The linearization of f at $x=\pi$ is then:

$$L(x) = -x+\pi$$

2) The linear function $L(x)$ gives an approximation of $f(x)$ near π:

$$f(x) \approx L(x) \qquad \text{near } \pi$$
$$\Downarrow$$
$$f(3) \approx L(3) = -3+\pi \sim -3 + 3.1415 = 0.1415$$

Note that the actual value of $\sin(3)$ given by a calculator is 0.1411...

Example 2

Find the linear approximation of the function $\sqrt{x+3}$ at $x = 1$ and use it to approximate the numbers $\sqrt{3.98}$ and $\sqrt{4.05}$

Solution

$$f(x) = \sqrt{x+3} \Rightarrow f'(x) = \frac{1}{2\sqrt{x+3}} \Rightarrow f'(1) = \frac{1}{2\sqrt{1+3}} = \frac{1}{4}$$

The tangent line to the graph of $f(x)$ at the point $(1, f(1)) = (1, 2)$ has equation:

$$y - f(1) = f'(1)(x-1)$$

$$y - 2 = \frac{1}{4}(x-1)$$

$$y = \frac{1}{4}(x-1) + 2$$

$$y = \frac{1}{4}x + \frac{7}{4}$$

The linearization of $f(x)$ at $x=1$ is the function $L(x)$:

$$L(x) = \frac{1}{4}x + \frac{7}{4}$$

$$\Downarrow$$

$$f(x) \approx L(x) \quad \text{near } 1$$

Note now that:

$$\sqrt{3.98} = \sqrt{3+0.98} = f(0.98)$$

$$\sqrt{4.05} = \sqrt{3+1.05} = f(1.05)$$

Then:

$$f(0.98) \approx L(0.98) = \frac{1}{4}(0.98) + \frac{7}{4} = 1.995$$

$$f(1.05) \approx L(1.05) = \frac{1}{4}(1.05) + \frac{7}{4} = 2.025$$

Note that the actual values of $\sqrt{3.98}$ and $\sqrt{4.05}$ given by a calculator are:

$$\sqrt{3.98} = 1.99499\ldots$$

$$\sqrt{4.05} = 2.01246\ldots$$