RELATED RATES (Sec. 2.7)

Before talking about related rates, let us recall what a (istantaneous) rate of change is.

In physics it is normal to meet with quantities that depend on other quantities (in particular time).

For example, when an object is moving, its position s changes with respect to time. This leads naturally to consider the istantaneous rate of change of the position with respect to time:

$$\frac{ds}{dt} \quad \begin{matrix} \to \text{dependent variable} \\ \to \text{independent variable} \end{matrix}$$

This is nothing else than the derivative of the position function $s(t)$ with respect to time and is normally referred as instantaneous velocity $v(t)$:

$$v(t) = s'(t) = \frac{ds}{dt}.$$

Now, also the velocity of an object can change with respect to time (if the object is speeding up or slowing down). So we can consider the rate of change of the velocity with respect to time, which is more commonly known as acceleration:

$$a(t) = v'(t) = \frac{dv}{dt} \left(= s''(t) = \frac{d^2 s}{dt^2} \right)$$

We will see that in most of the cases the rate of change is with respect to time. But there are examples of quantities that change with respect to quantities that are not time. For example, the pressure P of an object submerged in a fluid is:

$$P = \rho g h$$

where:

- ρ (rho) is the density of the fluid;
- g is the acceleration of gravity;
- h is the height of the fluid above the object (depth).

This means that the pressure experienced by an object submerged in a fluid changes with respect to the depth of the object. Then, if we consider the rate of change of the pressure with respect to depth we have

$$\frac{d}{dh} P = \frac{d}{dh} \underbrace{\rho g}_{\text{constant}} h$$

$$\Downarrow$$

$$\frac{dP}{dh} = \rho g \quad \text{which means that the rate of change is constant}$$

Now a related rates problem consists in computing the rate of change of one quantity in terms of the rate of change of another quantity (which is known or more easily measured).

CRUCIAL POINT: Find an equation that relates the quantities.

Indeed, by differentiating both sides of the equation, one gets an equation that relates the corresponding rates.

This will be clearer on the following examples.

EX. 1: Each side of a square is increasing at a rate of 6 cm/s. At what rate is the area of the square increasing when the area of the square is 16 cm²?

As in implicit differentiation, also in related rates we can proceed through steps:

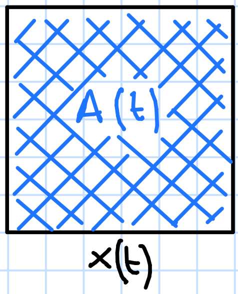
① Understand the problem, draw a picture, find and name the quantities which are related.

write the unit of measures

At a given time t let:

$x(t)$ = length of the side (cm)

$A(t)$ = area of the square (cm^2)



② Write what you know and what you wish to find.

Known: $\dfrac{dx}{dt} = 6 \dfrac{cm}{s}$; $A(t_0) = 16\ cm^2$

x measured in cm

t measured in seconds

this means that at a certain time $t = t_0$ the area is $16 cm^2$.

Unknown: $\left.\dfrac{dA}{dt}\right|_{t=t_0}$ ← this is another way of writing $A'(t_0)$

③ Find how the quantities are related (i.e. find a suitable equation which relates the quantities).

At each time t

$$A(t) = x^2(t) \longrightarrow$$

in particular this implies

$A(t_0) = x^2(t_0)$

$16 = x(t_0)$

$\Downarrow$

$x(t_0) = 4$ cm

④ Differentiate the above equation (so that the related quantities will give you the related rates).

IMPLICIT DIFFERENTIATION

$$\frac{d}{dt} A(t) = \frac{d}{dt} x^2(t)$$

$$\frac{dA}{dt} = 2x(t)\frac{dx}{dt}$$

⑤ Solve for the unknown quantity and replace the known data (with units of measures).

it depends on the length of the side

$$\frac{dA}{dt} = 2x(t)\frac{dx}{dt}$$

evaluate the previous equation at $t = t_0$

$$\left.\frac{dA}{dt}\right|_{t=t_0} = 2x(t_0)\left.\frac{dx}{dt}\right|_{t=t_0}$$

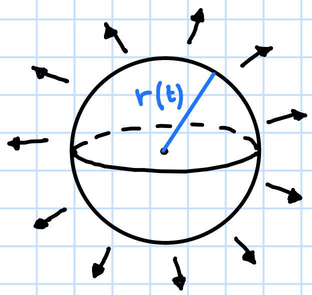
A measured in cm^2

$$\left.\frac{dA}{dt}\right|_{t=t_0} = 2\cdot(4\ cm)\cdot\left(6\ \frac{cm}{s}\right) = 48\ \frac{cm^2}{s}$$

time measured in seconds

EX.2 : Air is being pumped into a spherical ballon so that its volume increases at a rate of $100\ cm^3/s$.
How fast is the radius of the ballon increasing when the diameter is $50\ cm$?

① PICTURE & VARIABLES



At a given time t:

$r(t)$: radius of the ballon (cm)

$V(t)$: volume of the ballon (cm^3)

② KNOWN/UNKNOWN

Known : $\frac{dV}{dt} = 100\ \frac{cm^3}{s}$; $2r(t_0) = 50\,cm \Rightarrow r(t_0) = 25\,cm$

Unknown : $\left.\frac{dr}{dt}\right|_{t=t_0}$

③ EQUATION

At each time t

$$V(t) = \frac{4}{3}\pi r^3(t)$$ → formula of the volume of a sphere.

④ DIFFERENTIATE

$$\frac{d}{dt} V(t) = \frac{d}{dt} \frac{4}{3}\pi r^3(t)$$
$$\Downarrow$$
$$\frac{dV}{dt} = \frac{4}{3}\pi \frac{d}{dt} r^3(t)$$
$$\Downarrow$$
$$\frac{dV}{dt} = \frac{4}{3}\pi \cdot 3r^2(t) \cdot \frac{dr}{dt}$$

⑤ SOLVE AND SUBSTITUTE

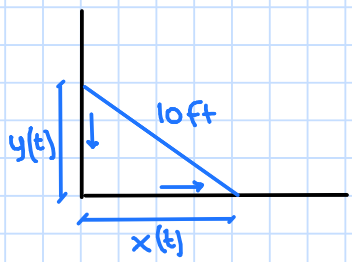
$$\left.\frac{dr}{dt}\right|_{t=t_0} = \left.\frac{dV}{dt}\right|_{t=t_0} \cdot \frac{1}{4\pi r^2(t_0)} = 100 \cdot \frac{cm^3}{s} \cdot \frac{1}{4\pi \cdot (25\,cm)^2} = \frac{1}{25 \cdot \pi} \cdot \frac{cm}{s}$$

$$\frac{dV}{dt} = 100 \frac{cm^3}{s}$$

$$r(t_0) = 25 \text{ cm}$$

EX. 3: A ladder 10 ft long rests against a vertical wall. If the bottom of the ladder slides away from the wall at a rate of 1 ft/s, how fast is the top of the ladder sliding down the wall when the bottom of the ladder is 6 ft from the wall?

① PICTURE / VARIABLES



At a given time t

$x(t)$: distance from the bottom of the ladder to the wall (ft)

$y(t)$: distance from the top of the ladder to the grond (ft)

② KNOWN / UNKNOWN

Known : $\frac{dx}{dt} = 1 \frac{ft}{s}$, $x(t_0) = 6$ ft

Unknown : $\left.\frac{dy}{dt}\right|_{t=t_0}$

③ EQUATION

At each time t:

$$x^2(t) + y^2(t) = 10^2 \leftarrow \text{Pythagorean theorem}$$

④ DIFFERENTIATE

$$\frac{d}{dt}\left[x^2(t) + y^2(t)\right] = \frac{d}{dt} 100$$

$$\Downarrow$$

$$\frac{d}{dt} x^2(t) + \frac{d}{dt} y^2(t) = 0$$

$$\Downarrow$$

$$2x(t) \cdot \frac{dx}{dt} + 2y(t) \frac{dy}{dt} = 0$$

⑤ SOLVE / SUBSTITUTE

$$\left.\frac{dy}{dt}\right|_{t=t_0} = \frac{-2x(t_0) \cdot \left.\frac{dx}{dt}\right|_{t=t_0}}{2y(t_0)} = \frac{-2 \cdot 6 \cancel{ft} \cdot 1 \frac{ft}{s}}{2 \cdot 8 \cancel{ft}} = -\frac{12}{16} = -\frac{3}{4} \frac{ft}{s}$$

$x(t_0) = 6$ ft

$\frac{dx}{dt} = 1$ ft/s

$y(t_0) = \sqrt{10^2 - x^2(t_0)} = \sqrt{100 - 36} = \sqrt{64} = 8$ ft

the result is negative since the distance from the top of the ladder to the ground is decreasing

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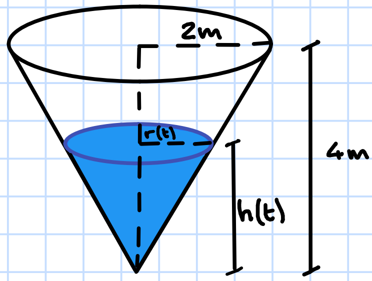
Ex. 4 : A water tank has the shape of an inverted circular cone with base radius 2m and height 4m. If water is being pumped into the tank at a rate of 2 m³/min, find the rate at which the water level is rising when the water is 3 m deep.

① PICTURE + VARIABLES

At a given time t:

- $V(t)$: volume of the water (m^3)
- $r(t)$: radius of the surface (m)
- $h(t)$: height of the water (m)



② KNOWN / UNKNOWN

Known: $\frac{dV}{dt} = 2\ \frac{m^3}{min}$, $h(t_0) = 3m$

Unknown: $\left.\frac{dh}{dt}\right|_{t=t_0}$

③ EQUATION

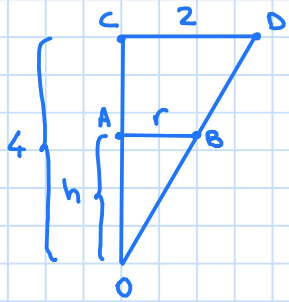
At each time t

$$V(t) = \frac{1}{3}\pi r^2(t)\, h(t).$$

Since we are looking for $\frac{dh}{dt}$, it would be useful to "eliminate" $r(t)$ from the previous equation.

For that we use the similar triangles $\hat{OAB} \sim \hat{OCD}$

$$\frac{h}{4} = \frac{r}{2} \Rightarrow r = \frac{h}{2}$$



So the equation becomes:

$$V(t) = \frac{1}{3}\pi \left(\frac{h(t)}{2}\right)^2 . h(t) = \frac{\pi}{12} h^3(t)$$

④ DIFFERENTIATE

$$\frac{d}{dt} V(t) = \frac{d}{dt} \frac{\pi}{12} h^3(t)$$

$$\Downarrow$$

$$\frac{dV}{dt} = \frac{\pi}{12} \cdot 3 \cdot h^2(t) \cdot \frac{dh}{dt}$$

⑤ SOLVE / SUBSTITUTE

$$\left.\frac{dh}{dt}\right|_{t=t_0} = \left.\frac{dV}{dt}\right|_{t=t_0} \cdot \frac{12}{\pi \cdot 3 \cdot h^2(t_0)} = 2\ \frac{m^3}{min} \cdot \frac{4}{\pi \cdot (3m)^2} = \frac{2 \cdot 4}{9\pi} \cdot \frac{m}{min} = \frac{8}{9\pi}\ \frac{m}{min}$$