# THE DERIVATIVE AS A FUNCTION (Sec. 2.2)

We ended the previous class with the following definition:

**Def**: The derivative of a function $f$ at a number $a$, denoted $f'(a)$, is:

$$f'(a) = \lim_{h\to 0} \frac{f(a+h)-f(a)}{h}$$

if this limit exists.

If now the number $a$ runs over the real numbers, we can replace it by the variable $x$ and we obtain:

$$f'(x) = \lim_{h\to 0} \frac{f(x+h)-f(x)}{h}$$

This is a function which is called the derivative function of $f$, since it has been "derived" from $f$.

Geometrically $f'(x)$ can be interpreted as the slope of the tangent line to the graph $y=f(x)$ at the point $(x, f(x))$.

Note that the domain of $f'(x)$ is given by the values $x$ at which the derivative is defined, i.e. the values at which the limit

$$\lim_{h\to 0} \frac{f(x+h)-f(x)}{h}$$

exists (left-hand limit = right-hand limit $= L$, where $L$ is not $\infty$ or $-\infty$).

And since for computing $f'(x)$ we need that $f$ is defined at $x$ ($f(x)$ appears in the limit) then we have:

$$\text{domain of } f'(x) \subseteq \text{domain of } f(x)$$

"is contained"

<u>ex</u>: Find the derivative function of $f(x) = x^2+3$

$$f'(x) = \lim_{h\to 0} \frac{f(x+h)-f(x)}{h} = \lim_{h\to 0} \frac{(x+h)^2+3-(x^2+3)}{h} =$$

$$= \lim_{h\to 0} \frac{\cancel{x^2}+2hx+h^2\cancel{+3}\cancel{-x^2}\cancel{-3}}{h} = \lim_{h\to 0} \frac{2hx+h^2}{h} =$$

$$\lim_{h\to 0} \frac{\cancel{h}(2x+h)}{\cancel{h}} = \lim_{h\to 0}(2x+h) = 2x$$

* Each time that we compute a derivative by using the definition we will simplify by $h$ in the last step

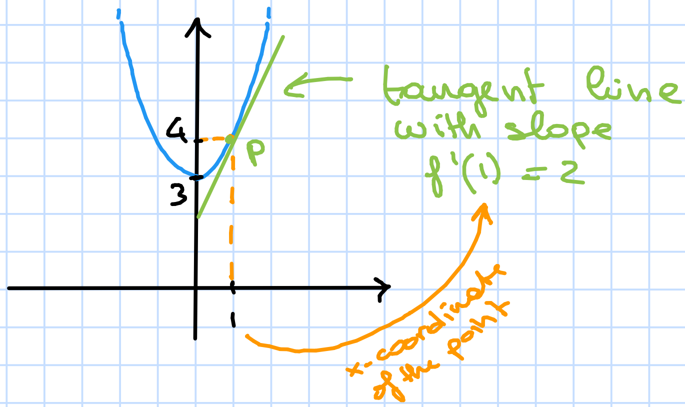
So we have: $f(x) = x^2 + 3 \implies f'(x) = 2x$

We can now compute the derivative of $f$ at each point by simply plugging in:

ex: $f'(1) = 2$

$\uparrow$

recall that this number represents the slope of the tangent line to the curve $y = f(x)$ at the point $P(1, f(1)) = (1,4)$




ex: Find the derivative of $f(x) = \sqrt{x}$.

$$f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h} = \lim_{h \to 0} \frac{\sqrt{x+h} - \sqrt{x}}{h} \cdot \frac{\sqrt{x+h} + \sqrt{x}}{\sqrt{x+h} + \sqrt{x}} =$$

$$= \lim_{h \to 0} \frac{x+h-x}{h(\sqrt{x+h} + \sqrt{x})} = \lim_{h \to 0} \frac{\cancel{h}}{\cancel{h}(\sqrt{x+h} + \sqrt{x})} =$$

$$= \lim_{h \to 0} \frac{1}{\sqrt{x+h} + \sqrt{x}} = \frac{1}{2\sqrt{x}}$$

So we have: $f(x) = \sqrt{x} \implies f'(x) = \frac{1}{2\sqrt{x}}$

Note that the domain of $f'(x)$, $D_{f'} = (0, \infty)$, is strictly contained in the domain of $f$, $D_f = [0, \infty)$.

We will say in this case that $f$ is "not differentiable at 0".

## NOTATION

There exist two fundamental notations for the derivative of a function $f$:

- Lagrange notation: $f'(x)$
- Leibniz notation: $\frac{df}{dx}$

We will use both of them depending on the context.

- LAGRANGE NOTATION (1770)

In Lagrange's notation a prime mark denotes a derivative

$f^{(\prime)}(x)$ this mark stays for "prime order"

Indeed we can compute higher order derivatives:

$f'(x)$ : first order derivative;

$f''(x) = (f'(x))'$ : second order derivative (= the derivative of the first order derivative)

$f'''(x) = (f''(x))'$ : third order derivative.

$f^{(4)}(x) = (f'''(x))'$

$\vdots$ → by iterating

$f^{(n)}(x) = (f^{(n-1)}(x))'$ : n-th order derivative (this is called in math a recurrene relation)

Hence, in general, we define the n-th derivative as the derivative of the (n-1)-th derivative

- LEIBNIZ NOTATION

This notation is particularly common when the equation $y = f(x)$ is regarded as a relationship between dependent (y) and independent variable (x):

$$\frac{df}{dx} = \frac{d}{dx} f(x) = \frac{dy}{dx}$$

dependent variable

independent variable

We remark that in this notation dx denotes an "infinitesimal" change in x, to which it corresponds an infinetesimal change dy in y:

so small that there is no way to measure it

$$\frac{dy}{dx} = \lim_{\Delta x \to 0} \frac{\Delta y}{\Delta x}.$$

With Leibniz notation, the value of the derivative at a number a will be denoted as follows

$$\left.\frac{df}{dx}\right|_{x=a} = \frac{df}{dx}(a) = \left.\frac{dy}{dx}\right|_{x=a} = \frac{dy}{dx}(a).$$

- OTHER NOTATIONS (less common): $Df(x)$, $D_x f(x)$.

We have seen that the domain of the derivative function $f'$ is not in general equal to the domain of $f$.

Indeed in the previous example $\left(f(x)=\sqrt{x} \Rightarrow f'(x)=\frac{1}{2\sqrt{x}}\right)$ we saw that zero is the domain of $f$ but not in the domain of $f'$.

We say that $f$ is not "differentiable" at 0.

Def: A function $f$ is differentiable at $a$ if $f'(a)$ exists, i.e. if

$$\lim_{h\to 0} \frac{f(a+h)-f(a)}{h} = L \text{, with } L \text{ a finite number (not } \infty \text{ or } -\infty).$$

$f$ is differentiable on an open interval if it is differentiable at every number in the interval.

Remark: $f$ is differentiable at $a$ if and only if $a$ is in the domain of $f'$.

Continuity and differentiability are desiderable properties for a function to have.
These two properties are related by the following theorem:

Theorem: If $f$ is differentiable at $a$ then $f$ is continuous at $a$

Proof: Let $f$ be a function that is differentiable at $a$.
Then, by definition, the limit $\lim_{x\to a} \frac{f(x)-f(a)}{x-a}$ exists, i.e.

$$\lim_{x\to a} \frac{f(x)-f(a)}{x-a} = L$$

with $L$ a number (not $\infty$ or $-\infty$).

We want to prove that $f$ is continuous at $a$, that is:

$$\lim_{x\to a} f(x) = f(a)$$

$\lim_{x\to a} f(x) = \lim_{x\to a} f(a)$

$\Downarrow$

$\lim_{x\to a} f(x) - \lim_{x\to a} f(a) = 0$

$\Updownarrow$ or equivalently

$$\lim_{x\to a} [f(x) - f(a)] = 0$$

We have :

multiply and divide by the same quantity; reorganize

$$\lim_{x\to a}[f(x)-f(a)] = \lim_{x\to a}[f(x)-f(a)]\cdot\frac{x-a}{x-a} = \lim_{x\to a}\frac{f(x)-f(a)}{x-a}\cdot(x-a)$$

$$\underset{\text{limit product law}}{=} \lim_{x\to a}\frac{f(x)-f(a)}{x-a}\cdot\lim_{x\to a}(x-a) = L\cdot 0 = 0 \checkmark$$

Hence $f$ is continuous at $a$. $\square$

in math we use a square for denoting the end of a proof

**Remarks** :

and this is also an implication :-)

- Recall that if the implication $P \Rightarrow Q$ is true then also the implication not $Q \Rightarrow$ not $P$ is true.

  So in this case the previous statement is equivalent to the following one:

  $f$ is not continuous at $a \Rightarrow f$ is not differentiable at $a$.

  This means that $f$ fails to be differentiable at each one of its discontinuities.

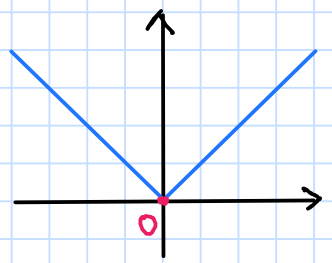
- The converse of this theorem is not true:

  $f$ is continuous at $a \not\Rightarrow f$ is differentiable at $a$

  does not imply

  For showing that this implication is not true we need a counterexample, i.e. an example of function which is continuous at a number $a$ but not differentiable at $a$.

<u>ex</u>: $f(x) = |x| = \begin{cases} -x & \text{when } x<0 \\ x & \text{when } x\geq 0 \end{cases}$ continuous at 0



slope of the line $y=-x$ is $-1$ $\Downarrow$ $\lim_{x\to 0^-}\frac{f(x)-f(0)}{x-0} = -1$

slope of the line $y=x$ is $1$ $\Downarrow$ $\lim_{x\to 0^+}\frac{f(x)-f(0)}{x-0} = 1$

Hence $\lim_{x\to 0^-}\frac{f(x)-f(0)}{x-0} \neq \lim_{x\to 0^+}\frac{f(x)-f(0)}{x-0} \Rightarrow f'(0) = \lim_{x\to 0}\frac{f(x)-f(0)}{x-0}$ does not exist

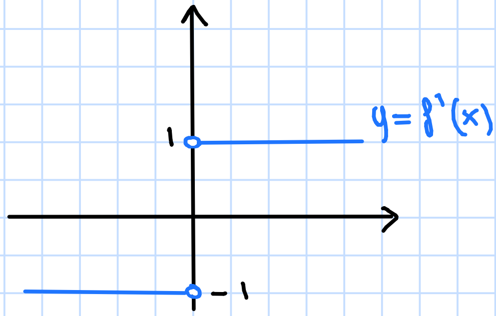
In conclusion $f(x)=|x|$ is continuous at 0, but not differentiable at 0

The derivative function of $f(x) = |x|$ is:

$$f'(x) = \begin{cases} -1 & \text{when } x < 0 \\ 1 & \text{when } x > 0 \end{cases}$$

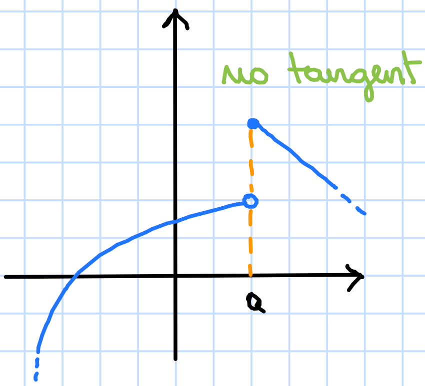
note that this function is not defined at 0

graph

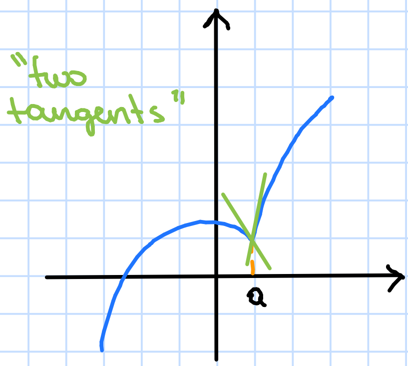



IN CONCLUSION:

## How can a function fail to be differentiable at a point?



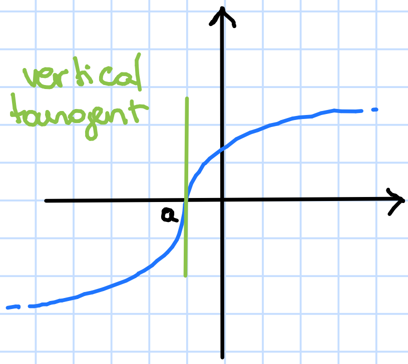

① $f$ is not continuous at $a$ ($a$ is a discontinuity)




② there is a "corner", "kink", cusp at $a$.

In this case we have actually two tangents that correspond to the different left-hand and right-hand limits

$$\lim_{x\to a^-} \frac{f(x)-f(a)}{x-a} \neq \lim_{x\to a^+} \frac{f(x)-f(a)}{x-a}$$




③ the tangent line is vertical at $a$.

In this case we have:

$$\lim_{x\to a^-} \frac{f(x)-f(a)}{x-a} = \infty \text{ or } -\infty$$

or

$$\lim_{x\to a^+} \frac{f(x)-f(a)}{x-a} = \infty \text{ or } -\infty$$

or

$$\lim_{x\to a} \frac{f(x)-f(a)}{x-a} = \infty \text{ or } -\infty$$

If you think to your function as a skate park recall that it is dangerous to skate on non differentiable functions!!