Calculus I - MAC 2311 - Section 003

Homework 2 - Solutions

Ex 1. Differentiate with respect to the indicated variable. If k appears in the function, treat it as a constant. Before starting computing your derivative, think if it is possible to simplify the function. Show all your work.

a) $$\frac{d}{dx}\left[\frac{x^{11}}{11}+\sqrt[5]{x}+\frac{1}{3x^2}+2\right]=\frac{d}{dx}\left[\frac{1}{11}x^{11}+x^{\frac{1}{5}}+\frac{1}{3}\cdot x^{-2}+2\right]=x^{10}+\frac{1}{5}x^{-\frac{4}{5}}-\frac{2}{3}x^{-3}.$$

b) $$\frac{d}{dt}\left[\sqrt[3]{\frac{1}{t^5}}\right]=\frac{d}{dt}\left[\left(t^{-5}\right)^{\frac{1}{3}}\right]=\frac{d}{dt}\left[t^{-\frac{5}{3}}\right]=-\frac{5}{3}t^{-\frac{5}{3}-1}=-\frac{5}{3}t^{-\frac{8}{3}}.$$

c) $$\frac{d}{du}\left[5u\tan(u)\right]\overset{\text{product rule}}{=}\frac{d}{du}\left[5u\right]\cdot\tan(u)+5u\frac{d}{du}\left[\tan(u)\right]=5\tan(u)+5u\sec^2(u).$$

d) $$\frac{d}{dx}\left[\frac{e^x+1}{\sin(3x)}\right]\overset{\text{quotient rule}}{=}\frac{\frac{d}{dx}(e^x+1)\cdot\sin(3x)-(e^x+1)\cdot\frac{d}{dx}(\sin(3x))}{(\sin(3x))^2}=$$
$$=\frac{e^x\cdot\sin(3x)-(e^x+1)\cdot\cos(3x)\cdot 3}{(\sin(3x))^2}.$$

e) $$\frac{d}{dx}\left[\frac{e^\pi+1}{\cos(3\pi)}\right]=0,\ \text{since}\ \frac{e^\pi+1}{\cos(3\pi)}\ \text{is a constant.}$$

f) $$\frac{d}{dt}\left[\frac{t\ln(t)+t}{t}\right]=\frac{d}{dt}\left[\frac{t\left(\ln(t)+1\right)}{t}\right]\overset{t\neq 0}{=}\frac{d}{dt}\left[\ln(t)+1\right]=\frac{1}{t}$$

g) $$\frac{d}{dx}\left[e^{x^2+1}+\ln(\cos(x))\right]\overset{\text{sum rule}}{=}\frac{d}{dx}\left[e^{x^2+1}\right]+\frac{d}{dx}\left[\ln(\cos(x))\right]\overset{\text{chain rule}}{=}$$
$$=e^{x^2+1}\cdot(x^2+1)'+\frac{1}{\cos(x)}\cdot(\cos(x))'=e^{x^2+1}\cdot 2x+\frac{1}{\cos(x)}\cdot(-\sin(x))=2x\cdot e^{x^2+1}-\tan(x).$$

h) $$\frac{d}{d\theta}\left[\cos\left(\pi\sqrt{\theta}\right)\right]\overset{\text{chain rule}}{=}-\sin\left(\pi\sqrt{\theta}\right)\cdot\left(\pi\sqrt{\theta}\right)'=-\sin\left(\pi\sqrt{\theta}\right)\cdot\frac{\pi}{2\sqrt{\theta}}.$$

i) $$\frac{d}{d\alpha}\left[\sqrt{\tan(k\alpha^2)}\right]\overset{\text{chain rule 1}}{=}\frac{1}{2\sqrt{\tan(k\alpha^2)}}\cdot\left(\tan(k\alpha^2)\right)'\overset{\text{chain rule 2}}{=}$$
$$=\frac{1}{2\sqrt{\tan(k\alpha^2)}}\cdot\sec^2(k\alpha^2)\cdot(k\alpha^2)'=\frac{\sec^2(k\alpha^2)\cdot 2k\alpha}{2\sqrt{\tan(k\alpha^2)}}$$

j) $$\frac{d}{dx}\left[e^{\ln(\sin(x))}\right]\overset{e^{\ln(x)}=x}{=}\frac{d}{dx}\left[\sin(x)\right]=\cos(x).$$

Note that the cancellation equation $e^{\ln(x)}=x$ is true for all $x>0$. Thus, if we want to be precise, we should point out that the simplification $e^{\ln(\sin(x))}=\sin(x)$ is true when $\sin(x)>0$. As a consequence $\cos(x)$ is the derivative of $e^{\ln(\sin(x))}$ for all x such that $\sin(x)>0$ (note again that, since the function $e^{\ln(\sin(x))}$ is not defined when $\sin(x)\leq 0$, it is not differentiable at those points).

k) $\frac{d}{dt}\left[\ln\left((\sin(t))^{3k}\right)\right] \overset{\ln(x^r)=r\cdot\ln(x)}{=} \frac{d}{dt}\left[3k\cdot\ln\left(\sin(t)\right)\right] = 3k\cdot\frac{d}{dt}\left[\ln\left(\sin(t)\right)\right] =$

$= 3k\cdot\frac{1}{\sin(t)}\cdot\cos(t) = 3k\cot(t).$

l) $\frac{d}{dx}\left[x^{\cos(x)}\right]$

- ⋆ **I method** : Logarithmic differentiation

$$\begin{aligned}
y &= x^{\cos(x)} \\
\ln(y) &= \ln\left(x^{\cos(x)}\right) = \cos(x)\cdot\ln(x) \\
\frac{d}{dx}\left[\ln(y)\right] &= \frac{d}{dx}\left[\cos(x)\cdot\ln(x)\right] \\
\frac{1}{y}\cdot\frac{dy}{dx} &= -\sin(x)\ln(x) + \frac{\cos(x)}{x} \\
\frac{dy}{dx} &= y\cdot\left(-\sin(x)\ln(x) + \frac{\cos(x)}{x}\right) \\
\frac{dy}{dx} &= x^{\cos(x)}\cdot\left(-\sin(x)\ln(x) + \frac{\cos(x)}{x}\right).
\end{aligned}$$

- ⋆ **II method**

By using the identity $e^{\ln(x)} = x$, we can rewrite the function in the following way:

$$f(x) = x^{\cos(x)} = e^{\ln\left(x^{\cos(x)}\right)} = e^{\cos(x)\ln(x)}.$$

Hence we have:

$$\begin{aligned}
f'(x) &= \left(e^{\cos(x)\ln(x)}\right)' = \\
&= e^{\cos(x)\ln(x)}\left(\cos(x)\ln\left(x\right)\right)' = \\
&= e^{\cos(x)\ln(x)}\left(-\sin(x)\ln(x) + \frac{\cos(x)}{x}\right) = \\
&= x^{\cos(x)}\left(-\sin(x)\ln(x) + \frac{\cos(x)}{x}\right).
\end{aligned}$$

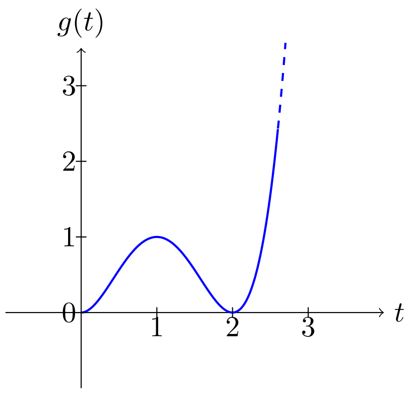
Ex 2. At a time $t = 0$ a calculus student leaves his home and starts walking toward the university where he has to take his calculus test. At some point he realizes that he has forgotten his calculator at home...

Assume that the student walks/runs according to the position function:

$$g(t) = t^4 - 4t^3 + 4t^2,$$

where t is in minutes and $g(t)$ in yards.

a) Find the velocity of the student as a function of t.
b) At what time(s) does he stop?
c) Find the acceleration of the student as a function of t.
d) Find his acceleration at $t = 3$ min.
e) Here to the right is the graph of the position function $g(t)$. Is your answer for (b) consistent with this graph? Why?



Solution:

a) We have:

$$v(t) = g'(t) = (t^4 - 4t^3 + 4t^2)' = 4t^3 - 12t^2 + 8t.$$

b) We have to find the time(s) $t > 0$ at which the velocity is 0, i.e. we have to solve:

$$v(t) = 0$$
$$\Downarrow$$
$$4t(t^2 - 3t + 2) = 0$$
$$\Downarrow$$
$$t(t-1)(t-2) = 0$$
$$\Downarrow$$
$$t = 0 \text{ min}, \text{ or } t = 1 \text{ min}, \text{ or } t = 2 \text{ min}.$$

Since we consider only the positive solutions (t >0), we have that the students stops at $t = 1$ min and $t = 2$ min.

c) We have:

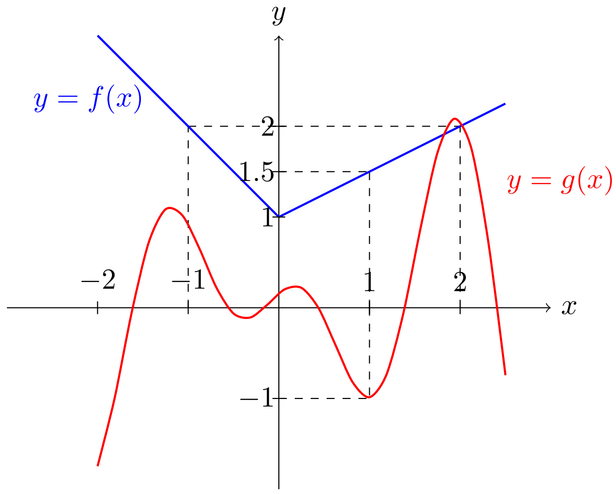
$$a(t) = v'(t) = (4t^3 - 12t^2 + 8t)' = 12t^2 - 24t + 8.$$

d) We have to compute $a(3)$:

$$a(3) = 12 \cdot 3^2 - 24 \cdot 3 + 8 = 108 - 72 + 8 = 44 \,\text{yards/min}^2$$

e) Yes, because geometrically the instantaneous velocity of the student at a time t is the slope of the tangent line to the graph of the position function at the point $(t, g(t))$. On the graph we clearly see that the tangent is horizontal at times $t = 1$ and $t = 2$.

Ex 3.



Let f and g be the functions whose graphs are shown above and let

$$h(x) = f(x) + g(x), \qquad u(x) = f(x)g(x), \qquad v(x) = \frac{f(x)}{g(x)}, \qquad w(x) = f(g(x)).$$

Compute $h'(1)$, $u'(1)$, $v'(1)$ and $w'(1)$, without finding explicit formulæ for $f(x)$ and $g(x)$.

Solution:

By using the differentiation rules (respectively sum, product, quotient and chain rule) we have:

$$\begin{aligned}
h'(x) &= f'(x) + g'(x);\\
u'(x) &= f'(x)g(x) + f(x)g'(x);\\
v'(x) &= \frac{f'(x)g(x) - f(x)g'(x)}{(g(x))^2};\\
w'(x) &= f'(g(x))g'(x).
\end{aligned}$$

Hence, in order to compute $h'(1)$, $u'(1)$, $v'(1)$ and $w'(1)$, we need before to find the values for $f(1)$, $g(1)$, $f'(1)$, $g'(1)$, $f'(g(1))$.

- Easily from the graphs of f and g we get that $f(1) = 1.5$ and $g(1) = -1$.
- For computing $f'(1)$ (respectively $g'(1)$) we need to find the slope of the tangent line to the graph $y = f(x)$ (respectively $y = g(x)$) at the point $(1, f(1))$ (respectively $(1, g(1))$).

  In the first case, the graph $y = f(x)$ is a line, which is tangent to itself at each point. Thus, we can compute its slope by using the coordinates of two of its points, for example $(0, 1)$ and $(2, 2)$, and we have:

  $$f'(1) = \frac{2-1}{2-0} = 0.5.$$

  In the second case, the tangent line to $y = g(x)$ at $(1, g(1))$ is horizontal (parallel to the x-axis), so that its slope is 0. This means that

  $$g'(1) = 0.$$

- Finally we have:

$$f'(g(1)) = f'(-1) = \frac{2-1}{-1-0} = -1,$$

  where $f'(-1)$ is computed as the slope of the line passing through the points $(-1,2)$ and $(0,1)$.

We are now ready for computing $h'(1)$, $u'(1)$, $v'(1)$ and $w'(1)$:

$$\begin{aligned}
h'(1) &= f'(1) + g'(1) = 0.5 + 0 = 1;\\
u'(1) &= f'(1)g(1) + f(1)g'(1) = 0.5 \cdot (-1) + 1.5 \cdot 0 = -0.5;\\
v'(1) &= \frac{f'(1)g(1) - f(1)g'(1)}{(g(1))^2} = \frac{0.5 \cdot (-1) - 1.5 \cdot 0}{(-1)^2} = \frac{-0.5}{1} = -0.5;\\
w'(1) &= f'(g(1))g'(1) = -1 \cdot 0 = 0.
\end{aligned}$$

Ex 4. The **ideal gas law** relates the temperature, pressure, and volume of an ideal gas. Given n moles of gas, the pressure P (in kPa), volume V (in liters), and temperature T (in kelvin) are related by the equation

$$PV = nRT,$$

where R is the molar gas constant ($R \cong 8.314 \frac{\text{kPa} \cdot \text{ liters}}{\text{kelvin}}$). Assume that the pressure, the volume and the temperature of the gas depend all on time.

a) Suppose that one mole of ideal gas is held in a closed container with a volume of 25 liters. If the temperature of the gas is increasing at a rate of 3.5 kelvin/min, how quickly will the pressure increase?

b) Suppose instead that the temperature of the gas is held fixed at 300 kelvin, while the volume decreases at a rate of 2.0 liters/min. How quickly is the pressure of the gas increasing at the instant that the volume is 20 liters?

Solution:

For both parts (a) and (b) we can define the following variables:

At a given time t:

- $P(t)$: the pressure of the gas (kPa);
- $V(t)$: the volume of the gas (L);
- $T(t)$: the temperature of the gas (K).

These variables are related by the ideal gas law:

$$P(t)V(t) = nRT(t),$$

where n is number of moles of gas and $R = 8.314 \frac{\text{kPa} \cdot L}{K}$.

In order to find how the corresponding rates are related, we differentiate the previous equation both sides:

$$\frac{d}{dt}\left(P(t)V(t)\right) = \frac{d}{dt}\left(nRT(t)\right)$$
$$\Downarrow \text{product rule}$$
$$\frac{dP}{dt}\cdot V(t) + P(t)\cdot\frac{dV}{dt} = nR\cdot\frac{dT}{dt}$$

At this point the situations described in (a) and in (b) are different and will be analyzed separately.

a)
- **Known:**
  - $V(t) = 25$ L for all t (the volume is constant) $\Rightarrow \dfrac{dV}{dt} = 0$;
  - $\dfrac{dT}{dt} = 3.5\frac{K}{\text{min}}$ for all t.
  - $n = 1$; $R = 8.314\frac{\text{kPa}\cdot L}{K}$.
- **Unknown:** $\dfrac{dP}{dt}$.

By replacing the known data in

$$\frac{dP}{dt}\cdot V(t) + P(t)\cdot\frac{dV}{dt} = nR\cdot\frac{dT}{dt}$$

we obtain:

$$\frac{dP}{dt}\cdot 25 + P(t)\cdot 0 = 1\cdot 8.314\cdot 3.5$$
$$\Downarrow$$
$$\frac{dP}{dt} = \frac{8.314\cdot 3.5}{25} = 1.16396\,\frac{\text{kPa}}{\text{min}}$$

b)
- **Known:**
  - $T(t) = 300$ K for all t (the temperature is constant) $\Rightarrow \dfrac{dT}{dt} = 0$;
  - $\dfrac{dV}{dt} = -2\frac{\text{L}}{\text{min}}$ for all t.
  - there is a time t_0 such that $V(t_0) = 20$ L;
  - $n = 1$; $R = 8.314\frac{\text{kPa}\cdot L}{K}$.
- **Unknown:** $\left.\dfrac{dP}{dt}\right|_{t=t_0}$.

By evaluating at $t = t_0$ the equation

$$\frac{dP}{dt}\cdot V(t) + P(t)\cdot\frac{dV}{dt} = nR\cdot\frac{dT}{dt}$$

we obtain:

$$\left.\frac{dP}{dt}\right|_{t=t_0} \cdot V(t_0) + P(t_0) \cdot \left.\frac{dV}{dt}\right|_{t=t_0} = nR \cdot \left.\frac{dT}{dt}\right|_{t=t_0}$$
$$\Downarrow$$
$$\left.\frac{dP}{dt}\right|_{t=t_0} \cdot 20 + P(t_0) \cdot (-2) = nR \cdot 0$$
$$\Downarrow$$
$$\left.\frac{dP}{dt}\right|_{t=t_0} = \frac{2P(t_0)}{20}$$

In order to find $P(t_0)$ we use the ideal gas law:

$$P(t_0)V(t_0) = nRT(t_0)$$
$$\Downarrow$$
$$P(t_0) = \frac{nRT(t_0)}{V(t_0)} = \frac{1 \cdot 8.314 \cdot 300}{20} = 124.71\,\text{kPa}$$

In conclusion we have:

$$\left.\frac{dP}{dt}\right|_{t=t_0} = \frac{2P(t_0)}{20} = \frac{2 \cdot 124.71}{20} = 12.471\,\frac{\text{kPa}}{\text{min}}.$$

Ex 5. Which statements are True/False? Justify your answers.

a) If $f(x)$ is a polynomial of degree n then $f^{(n+1)}(x) = 0$.

True. If $f(x)$ is a polynomial of degree n, then $f'(x)$ is a polynomial of degree $n-1$, $f''(x)$ is a polynomial of degree $n-2, \dots, f^{(n)}(x)$ is a polynomial of degree 0, i.e. $f^{(n)}(x)$ is a constant. Therefore $f^{(n+1)}(x) = \left(f^{(n)}(x)\right)' = 0$.

b) Let $F(t)$ be a physical quantity depending on time. If $\frac{dF}{dt}$ is constant for each time t, then F is constant.

False. If $F(t) = t$, then F is a non constant function with $\frac{dF}{dt} = 1$ for all t.

c) Let $h(x) = g(f(x))$. If $f'(0) = 1$ and $g'(0) = 0$, then $h'(0) = 0$.

False. In order to show that the statement is false, we have to provide a **counterexample**, i.e. an example of two functions $f(x)$ and $g(x)$ such that $f'(0) = 1$ and $g'(0) = 0$, but $h'(0) \neq 0$.

Let us consider $f(x) = x + 1$ and $g(x) = x^2$. We have $f'(x) = 1$ and $g'(x) = 2x$, therefore $f'(0) = 1$ and $g'(0) = 0$. Now $h(x) = g(f(x)) = (x+1)^2$, so $h'(x) = 2(x+1)$ and $h'(0) = 2(0+1) = 2 \neq 0$.

d) We have $\ln(3e^2) - \ln(3\sqrt{e}) = \frac{3}{2}$.

True. Indeed we have:

$$\ln(3e^2) - \ln(3\sqrt{e}) = \ln\left(\frac{3e^2}{3\sqrt{e}}\right) = \ln\left(\frac{e^2}{e^{\frac{1}{2}}}\right) = \ln\left(e^{2-\frac{1}{2}}\right) = \ln\left(e^{\frac{3}{2}}\right) = \frac{3}{2}$$