Calculus I - MAC 2311 - Section 003
Homework 1 - Solutions

Ex 1. Compute the following limits and show all your work:

a) $\lim_{x\to 2}\frac{\sin(\pi x)}{x+1}\overset{\text{plug in}}{=}\frac{\sin(2\pi)}{2+1}=\frac{0}{3}=0.$

b) $\lim_{t\to 3}\frac{t^2-2t-3}{2t-6}=\lim_{t\to 3}\frac{(t-3)(t+1)}{2(t-3)}=\lim_{t\to 3}\frac{t+1}{2}\overset{\text{plug in}}{=}\frac{3+1}{2}=2.$

c) $\lim_{x\to 0}\frac{\sqrt{x+1}-1}{x}=\lim_{x\to 0}\frac{\sqrt{x+1}-1}{x}\cdot\frac{\sqrt{x+1}+1}{\sqrt{x+1}+1}=\lim_{x\to 0}\frac{x+1-1}{x\left(\sqrt{x+1}+1\right)}=$

$=\lim_{x\to 0}\frac{x}{x\left(\sqrt{x+1}+1\right)}=\lim_{x\to 0}\frac{1}{\sqrt{x+1}+1}\overset{\text{plug in}}{=}\frac{1}{1+1}=\frac{1}{2}.$

d) $\lim_{x\to\infty}\frac{\pi x^7+2x-1}{-3x^7+x^5}=\lim_{x\to\infty}\frac{x^7\left(\pi+\frac{2}{x^6}-\frac{1}{x^7}\right)}{x^7\left(-3+\frac{1}{x^2}\right)}=\lim_{x\to\infty}\frac{\pi+\frac{2}{x^6}-\frac{1}{x^7}}{-3+\frac{1}{x^2}}=“\frac{\pi+\frac{2}{\infty}-\frac{1}{\infty}}{-3+\frac{1}{\infty}}”=$

$=\frac{\pi+0-0}{-3+0}=-\frac{\pi}{3}.$

e) $\lim_{u\to-\infty}\frac{-u^3+3u}{u+1}=\lim_{u\to-\infty}\frac{u^3\left(-1+\frac{3}{u^2}\right)}{u\left(1+\frac{1}{u}\right)}=\lim_{u\to-\infty}\frac{u^2\left(-1+\frac{3}{u^2}\right)}{1+\frac{1}{u}}=“\frac{(-\infty)^2\cdot\left(-1+\frac{3}{(-\infty)^2}\right)}{1+\frac{1}{-\infty}}”=$

$“\frac{\infty\cdot(-1+0)}{1+0}”=“\frac{\infty\cdot(-1)}{1}”=-\infty.$

f) $\lim_{t\to\infty}\frac{t+5}{2t^5-3t^3-1}=\lim_{t\to\infty}\frac{t\left(1+\frac{5}{t}\right)}{t^5\left(2-\frac{3}{t^2}-\frac{1}{t^5}\right)}=\lim_{t\to\infty}\frac{1+\frac{5}{t}}{t^4\left(2-\frac{3}{t^2}-\frac{1}{t^5}\right)}=“\frac{1+\frac{5}{\infty}}{\infty\cdot\left(2-\frac{3}{\infty}-\frac{1}{\infty}\right)}”=$

$“\frac{1+0}{\infty\cdot(2-0-0)}”=“\frac{1}{\infty\cdot 2}”=“\frac{1}{\infty}”=0.$

g) $\lim_{\alpha\to 0}\frac{\sin(2018\alpha)}{2019\alpha}=\lim_{\alpha\to 0}\frac{1}{2019}\cdot\frac{\sin(2018\alpha)}{\alpha}\cdot\frac{2018}{2018}=\lim_{\alpha\to 0}\frac{2018}{2019}\cdot\frac{\sin(2018\alpha)}{2018\alpha}=$

$=\frac{2018}{2019}\cdot\lim_{\alpha\to 0}\frac{\sin(2018\alpha)}{2018\alpha}\overset{\lim_{x\to 0}\frac{\sin x}{x}=1}{=}\frac{2018}{2019}\cdot 1=\frac{2018}{2019}$

h) $\lim_{\theta\to\frac{\pi}{2}^+}\frac{\cos\theta-1}{\cos\theta}\overset{\text{plug in}}{=}\frac{\cos\left(\frac{\pi}{2}\right)-1}{\cos\left(\frac{\pi}{2}\right)}=\frac{0-1}{0}=“\frac{-1}{0}”$

Since we are computing a one-sided limit, this means that the result of the limit will be either ∞ or $-\infty$ depending on the sign of the denominator. In this case, we have that when θ approaches $\frac{\pi}{2}$ from the right (i.e. $\theta>\frac{\pi}{2}$) then $\cos(\theta)<0$ (in order to convince yourself draw on the unit circle an angle θ close to $\frac{\pi}{2}$ such that $\theta>\frac{\pi}{2}$, or think about the graph of the function cosine...). Thus:

$\lim_{\theta\to\frac{\pi}{2}^+}\frac{\cos\theta-1}{\cos\theta}=“\frac{-1}{0^-}”=\infty$

i) $\lim\limits_{x\to -1} \dfrac{x^2}{x+1} \overset{\text{plug in}}{=} \dfrac{(-1)^2}{-1+1} = \text{“}\dfrac{1}{0}\text{”}$

We will solve this limit by computing separately the left-hand and the right-hand limits:

$$\lim_{x\to(-1)^-} \frac{x^2}{x+1} \overset{x+1<0}{=} \text{“}\frac{1}{0^-}\text{”} = -\infty.$$

$$\lim_{x\to(-1)^+} \frac{x^2}{x+1} \overset{x+1>0}{=} \text{“}\frac{1}{0^+}\text{”} = +\infty.$$

Since $\lim\limits_{x\to(-1)^-} \dfrac{x^2}{x+1} \neq \lim\limits_{x\to(-1)^+} \dfrac{x^2}{x+1}$ then $\lim\limits_{x\to -1} \dfrac{x^2}{x+1}$ does not exist.

j) $\lim\limits_{x\to 2} f(x)$, where $\quad f(x) = \begin{cases} \frac{x^2-4}{x-2}, & \text{when } x < 2 \\ \sqrt{x+2}+2 & \text{when } x \geq 2 \end{cases}$

Since we have to compute the limit of a piecewise-defined function at its “breaking point” $x = 2$, we have first to compute separately the left-hand and the right-hand limits:

$$\lim_{x\to 2^-} f(x) \overset{x<2}{=} \lim_{x\to 2^-} \frac{x^2-4}{x-2} = \lim_{x\to 2^-} \frac{(x-2)(x+2)}{x-2} = \lim_{x\to 2^-} x+2 \overset{\text{plug in}}{=} 2+2 = 4.$$

$$\lim_{x\to 2^+} f(x) \overset{x\geq 2}{=} \lim_{x\to 2^+} \sqrt{x+2}+2 \overset{\text{plug in}}{=} \sqrt{2+2}+2 = 2+2 = 4.$$

Since $\lim\limits_{x\to 2^-} f(x) = \lim\limits_{x\to 2^+} f(x) = 4$ then $\lim\limits_{x\to 2} f(x) = 4$.

Ex 2. Sketch the graph of a function f which satisfies simultaneously the following conditions:

a) $\lim\limits_{x\to -\infty} f(x) = 0$,

b) f has a jump discontinuity at $x = -2$,

c) $f(-2) = 3$,

d) $\lim\limits_{x\to(-2)^+} f(x) = 3$,

e) $\lim\limits_{x\to 0^-} f(x) = -\infty$,

f) $x = 0$ is a solution for the equation $f(x) = 2$,

g) The line $y = 2$ is a horizontal asymptote.

Solution:

Let us translate some of these conditions into geometrical terms or in terms of limits.

a) $\lim\limits_{x\to -\infty} f(x) = 0$: this means that the line $y = 0$ is a horizontal asymptote for the graph of the function f.

b) f has a jump discontinuity at $x = -2$: this means that $\lim_{x\to -2^-} f(x) = L_1$ and $\lim_{x\to -2^+} f(x) = L_2$ with $L_1 \neq L_2$. In the example below we have $\lim_{x\to -2^-} f(x) = 1$ and $\lim_{x\to -2^+} f(x) = 3$ (the latter because of d)).

c) $f(-2) = 3$: the graph of the function passes through the point $(-2, 3)$.

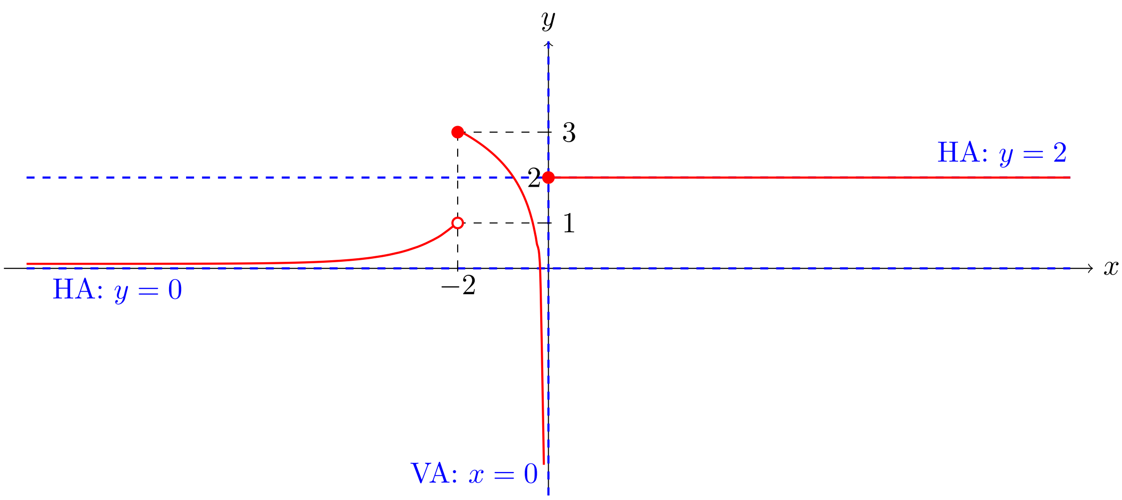
d) $\lim_{x\to(-2)^+} f(x) = 3$.

e) $\lim_{x\to 0^-} f(x) = -\infty$: this means that the line $x = 0$ is a vertical asymptote for the graph of the function f.

f) $x = 0$ is a solution for the equation $f(x) = 2$: this means that $f(0) = 2$, i.e. the graph of the function passes through the point $(0, 2)$.

g) The line $y = 2$ is a horizontal asymptote: this means that either $\lim_{x\to -\infty} f(x) = 2$ or $\lim_{x\to\infty} f(x) = 2$. Since we know already from a) that $\lim_{x\to -\infty} f(x) = 0$ (and the limit is unique) then we get $\lim_{x\to\infty} f(x) = 2$.

Of course there exist infinitely many examples of functions satisfying simultaneously all the previous conditions. An example is given by the function whose graph is the following. Note that constant functions have a horizontal asymptote.



Ex 3. Let f be the function defined as:

$$f(x) = \begin{cases} c^2 \cdot \cos(x+1) + 2c, & \text{when } x < -1 \\ \frac{c}{x+3} & \text{when } x \geq -1 \end{cases},$$

where c is a constant (i.e. a real number).

a) Compute $\lim_{x\to(-1)^-} f(x)$, $\lim_{x\to(-1)^+} f(x)$ and $f(-1)$.

b) Find the value(s) of c what make f continuous at $x = -1$.

c) If c is one of the values found in (b), is f continuous for all real numbers?

Solution:

We remark that $f(x)$ is a piecewise-defined function whose branches are respectively defined on the intervals $(-\infty,-1)$ and $[-1,\infty)$.

a)
- When $x<-1$ then $f(x)=c^2\cdot\cos(x+1)+2c$, hence:
  $\lim_{x\to-1^-} f(x)=\lim_{x\to-1^-} c^2\cdot\cos(x+1)+2c=c^2\cdot\cos(-1+1)+2c=c^2+2c$.
- When $x>-1$ then $f(x)=\frac{c}{x+3}$, hence:
  $$\lim_{x\to(-1)^+} f(x)=\lim_{x\to(-1)^+}\frac{c}{x+3}=\frac{c}{-1+3}=\frac{c}{2}.$$
- When $x=-1$ then $f(x)=\frac{c}{x+3}$, hence:
  $f(-1)=\frac{c}{-1+3}=\frac{c}{2}$.

b) By definition, f is continuous at -1 if and only if

$$\lim_{x\to-1^-} f(x)=\lim_{x\to-1^+} f(x)=f(-1)$$
$$\Updownarrow$$
$$c^2+2c=\frac{c}{2}$$
$$\Updownarrow$$
$$c^2+2c-\frac{c}{2}=0$$
$$\Updownarrow$$
$$2c^2+4c-c=0$$
$$\Updownarrow$$
$$2c^2+3c=0$$
$$\Updownarrow$$
$$c\cdot(2c+3)=0$$
$$\Updownarrow$$
$$c=0 \text{ or } 2c+3=0$$
$$\Updownarrow$$
$$c=0 \text{ or } c=-\frac{3}{2}$$

Therefore f is continuous at -1 if and only if $c=0$ or $c=-\frac{3}{2}$.

c) Let $c=0$ or $c=-\frac{3}{2}$. From b) we know that for these values of c, f is continuous at -1. So, in order to answer the question we have to study the continuity of f on the intervals $(-\infty,-1)$ and $(1,\infty)$.

On $(-\infty,-1)$ we have $f(x)=c^2\cdot\cos(x+1)+2c$, which is a continuous function everywhere ($\cos(x)$ is a continuous function).

On $(-1,\infty)$, the function f is defined as the rational function $\frac{c}{x+3}$, which is continuous on $(-1,\infty)$. Indeed the only value that vanishes the denominator is $x=-3$ and, since $-3<-1$, it does not belong to the interval $(-1,\infty)$.

Then YES!, for $c=0$ or $c=-\frac{3}{2}$ the function f is continuous for all real numbers.

Ex 4. Which statements are True/False? Justify your answers.

a) The function $f(x) = \frac{x^2-9}{x+3}$ has a vertical asymptote at $x = -3$.

False. By definition, $x = -3$ would be a vertical asymptote for $f(x)$ if either $\lim_{x\to -3^-} f(x) = \pm\infty$ or $\lim_{x\to -3^+} f(x) = \pm\infty$. In this case we have:

$$\lim_{x\to -3} f(x) = \lim_{x\to -3} \frac{x^2-9}{x+3} = \lim_{x\to -3} \frac{(x-3)(x+3)}{x+3} = \lim_{x\to -3} \frac{x-3}{1} = -6.$$

This means that $\lim_{x\to -3^-} f(x) = \lim_{x\to -3^+} f(x) = -6$, i.e. both the left-hand and the right-hand limits are finite. Hence $x = -3$ is not a vertical asymptote.

b) Let f be a function which is continuous at $x = 2$. If $\lim_{x\to 2} f(x) = 3$, then $f(2) = 3$.

True. Since f is continuous at $x = 2$, then by definition $\lim_{x\to 2} f(x) = f(2)$. Therefore, if $\lim_{x\to 2} f(x) = 3$, then $f(2) = 3$.

c) If f is a continuous function on $[a, b]$ such that $f(a) < 0$ and $f(b) > 0$ then the equation $f(x) = 0$ has at least a solution.

True. Since f is continuous on $[a, b]$ and $f(a) < 0 < f(b)$, then, by the Intermediate Value Theorem, there exists c in (a, b) such that $f(c) = 0$, which is a solution for the equation $f(x) = 0$.

d) There exists a rational function that has 2 different horizontal asymptotes.

False. Indeed, we will prove that a rational function has at most one horizontal asymptote. For it, we analyze three different cases.

Let $f(x) = \frac{P(x)}{Q(x)}$ be a rational function, where $P(x)$ and $Q(x)$ are polynomials. The following three cases may occur:

- $\deg(P) < \deg(Q)$: in this case $\lim_{x\to\pm\infty} f(x) = \lim_{x\to\pm\infty} \frac{P(x)}{Q(x)} = 0$. Therefore in this case $f(x)$ has a unique horizontal asymptote given by $y = 0$.
- $\deg(P) = \deg(Q)$: in this case $\lim_{x\to\pm\infty} f(x) = \lim_{x\to\pm\infty} \frac{P(x)}{Q(x)} = L$, where L is the ratio of the leading coefficients of P and Q. Therefore in this case $f(x)$ has an unique horizontal asymptote given by $y = L$.
- $\deg(P) > \deg(Q)$: in this case $\lim_{x\to\pm\infty} f(x) = \lim_{x\to\pm\infty} \frac{P(x)}{Q(x)} = \infty$ or $-\infty$. Therefore in this case $f(x)$ has no horizontal asymptote.

We conclude that a rational function has **at most** one horizontal asymptote.