Continuity

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We recall that a function $f : \mathbb{R} \to \mathbb{R}$ is said to be *continuous* at $a \in \mathbb{R}$ if

$$\lim_{x \to a} f(x) = f(a).$$

This means that f is continuous at a if and only if the following three conditions are satisfied:

1) $f(a)$ is defined (i.e. a is in the domain of the function);
2) $\lim_{x \to a} f(x)$ exists (i.e. $\lim_{x \to a^-} f(x) = \lim_{x \to a^+} f(x)$ and the limit is finite);
3) $\lim_{x \to a} f(x) = f(a)$.

If f is not continuous at a we say that f is *discontinuous* at a.

There exist three kinds of discontinuity:

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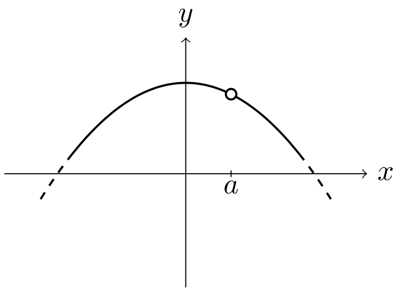
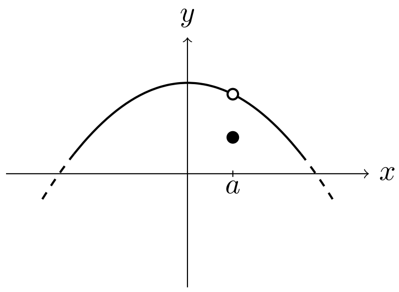
a is said to be a *removable discontinuity* if $\lim_{x \to a^-} f(x) = \lim_{x \to a^+} f(x) = L < \infty$ and either a does not belong to the domain of f or $f(a) \neq L$.



a does not belong to the domain



$f(a) \neq L$

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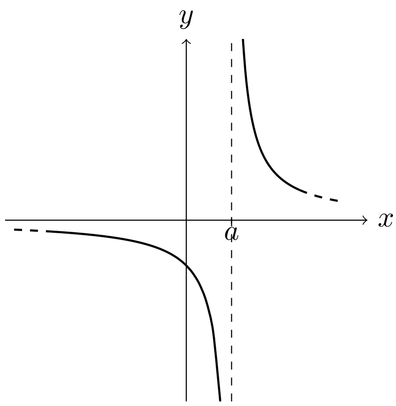
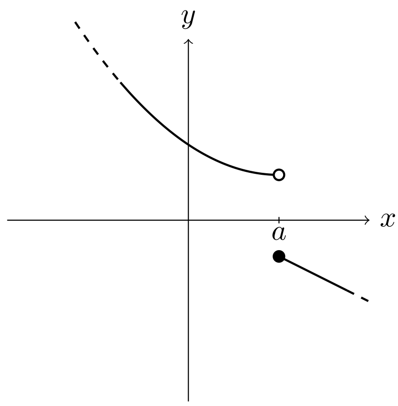
a is said to be an *infinite discontinuity* if $\lim_{x \to a^-} f(x) = \pm\infty$ or $\lim_{x \to a^+} f(x) = \pm\infty$.



a is said to be a *jump discontinuity* if $\lim_{x \to a^-} f(x) \neq \lim_{x \to a^+} f(x)$.



Examples.

1) Let us consider the function
$$f(x) = \frac{3+x}{x^4+3x^3}.$$
Locate its discontinuities and for each of them say if it is a removable, an infinite or a jump discontinuity.

Solution. First of all we can write
$$f(x) = \frac{3+x}{x^3(x+3)}.$$
Hence the domain of f is the set $D = \mathbb{R} \setminus \{-3, 0\}$.

Since $f(x)$ is a rational function, it is continuous at each point of its domain, so that we will just analyze the behavior of f at a neighborhood of $x = -3$ and $x = 0$. We have:

- $\lim_{x\to(-3)^-} \frac{3+x}{x^3(x+3)} = \lim_{x\to(-3)^-} \frac{1}{x^3} = -\frac{1}{27}$ and $\lim_{x\to(-3)^+} \frac{3+x}{x^3(x+3)} = \lim_{x\to(-3)^+} \frac{1}{x^3} = -\frac{1}{27}$. Hence $x = -3$ is a removable discontinuity.
- $\lim_{x\to 0^-} \frac{3+x}{x^3(x+3)} = \lim_{x\to 0^-} \frac{1}{x^3} = \text{“}\frac{1}{0^-}\text{”} = -\infty$ and $\lim_{x\to 0^+} \frac{3+x}{x^3(x+3)} = \lim_{x\to 0^+} \frac{1}{x^3} = \text{“}\frac{1}{0^+}\text{”} = \infty$. Hence $x = 0$ is an infinite discontinuity.

2) Let us consider the function
$$f(x) = \begin{cases} \frac{x-8}{2x-4}, & x < 0; \\ \sqrt{1-x}, & 0 \le x \le 1; \\ x-1, & x > 1 \end{cases}$$
Locate its discontinuities and for each of them say if it is a removable, an infinite or a jump discontinuity.

Solution. The function f is defined at all $x \in \mathbb{R}$. Since it is a piecewise function we need to analyse the behavior of f at its “junction points” $x = 0$ and $x = 1$. We have:

- $\lim_{x\to 0^-} f(x) = \lim_{x\to 0^-} \frac{x-8}{2x-4} = \frac{-8}{-4} = 2$ and $\lim_{x\to 0^+} f(x) = \lim_{x\to 0^+} \sqrt{1-x} = 1$. Hence $x = 0$ is a jump discontinuity.
- $\lim_{x\to 1^-} f(x) = \lim_{x\to 1^-} \sqrt{1-x} = 0$ and $\lim_{x\to 1^+} f(x) = \lim_{x\to 1^+} x-1 = 0$. Hence $\lim_{x\to 1^-} f(x) = = \lim_{x\to 1^+} f(x) = f(1) = 0$ so that f is continuous at $x = 1$.

In conclusion f is continuous at all $x \in \mathbb{R} \setminus \{0\}$.

Exercices.

1) For which values of x is the following function continuous? For each discontinuity establish if it is a removable, an infinite or a jump discontinuity.
$$f(x) = \frac{x-2}{x^2-3x+2}.$$
$x = 1$ infinite discontinuity
$x = 2$ removable discontinuity

2) For which values of x is the following function continuous? For each discontinuity establish if it is a removable, an infinite or a jump discontinuity.

$$f(x) = \frac{x}{|x|}.$$

$x = 0$ jump discontinuity

3) For which values of x is the following function continuous? For each discontinuity establish if it is a removable, an infinite or a jump discontinuity.

$$f(x) = \begin{cases} \frac{x^2-x}{x^2-1}, & x \neq 1; \\ 1, & x = 1 \end{cases}$$

$x = -1$ infinite discontinuity
$x = 1$ removable discontinuity

4) Determine all values of the real constant a so that the following function is continuous for all $x \in \mathbb{R}$.

$$f(x) = \begin{cases} ax^2 + x, & x < 2; \\ \frac{x^2-2a^2}{x}, & x \geq 2 \end{cases}$$

$a = 0$ or $a = -4$