Calculus I - MAC 2311 - Section 007

Quiz 6 - Solutions

11/02/2017

1) Compute the following limit:

$$\lim_{x\to 0} \left(x^3+1\right)^{\frac{1}{x^2}}.$$

Solution:

Let us set:

$$y = \left(x^3+1\right)^{\frac{1}{x^2}}.$$

We have:

$$\lim_{x\to 0} \left(x^3+1\right)^{\frac{1}{x^2}} = \lim_{x\to 0} y = \lim_{x\to 0} e^{\ln(y)} = e^{\lim_{x\to 0}\ln(y)}.$$

Thus, all we have to do is to compute $\lim_{x\to 0}\ln(y)$:

$$\lim_{x\to 0}\ln(y) = \lim_{x\to 0}\ln\left(\left(x^3+1\right)^{\frac{1}{x^2}}\right) = \lim_{x\to 0}\frac{1}{x^2}\ln\left(x^3+1\right) = \lim_{x\to 0}\frac{\ln\left(x^3+1\right)}{x^2}.$$

Now we have $\lim_{x\to 0}\ln\left(x^3+1\right) = 0$ and $\lim_{x\to 0} x^2 = 0$, so that we are faced with the indeterminate form $\frac{0}{0}$. Hence we can apply L'Hospital's Rule:

$$\lim_{x\to 0}\frac{\ln\left(x^3+1\right)}{x^2} = \lim_{x\to 0}\frac{\frac{1}{x^3+1}(3x^2)}{2x} = \lim_{x\to 0}\frac{3x^2}{2x(x^3+1)} = \lim_{x\to 0}\frac{3x}{2(x^3+1)} = \frac{0}{2} = 0.$$

Hence we get $\lim_{x\to 0}\ln(y) = 0$ so that

$$\lim_{x\to 0} \left(x^3+1\right)^{\frac{1}{x^2}} = e^{\lim_{x\to 0}\ln(y)} = e^0 = \mathbf{1}.$$

2) a) State Fermat's theorem.

Fermat's theorem
Let f be a function of domain D. If f has a local maximum or minimum at c in D and f is differentiable at c (i.e. $f'(c)$ exists) then $f'(c) = 0$.

b) Give the definition of a critical point of a function f.

A critical point (or number) *of a function f is a number c in the domain of f such that either $f'(c) = 0$ or $f'(c)$ does not exist.*

c) Find the absolute maximum and minimum values of the function

$$f(x) = -2x^3 - 3x^2 + 12x + 5$$

on the closed interval $[-3, 3]$.

Organize your solution in the following steps:

- Find the critical numbers of f and their corresponding values.
- Find the values of f at the endpoints of the interval $[-3, 3]$.

- Compare the values obtained in step 1 and step 2 and return the absolute maximum and the absolute minimum values of f.

Solution:

Since f is a continuous function and $[-3,3]$ a closed interval, the Extreme Value Theorem guarantees that f attains an absolute maximum value and an absolute minimum value on $[-3,3]$. Let us find them!

- *Find the critical numbers of f and their corresponding values.*

  Since f is a polynomial, it is differentiable on $\mathbb{R}$. Thus, its critical numbers are all the numbers c such that $f'(c) = 0$.
  Here we have:

  $$f'(x) = -6x^2 - 6x + 12 = -6(x^2 + x - 2) = -6(x+2)(x-1).$$

  Thus $f'(x) = 0$ if and only if $x = -2$ or $x = 1$. The corresponding values at $x = -2$ and $x = 1$ are $f(-2) = -15$ and $f(1) = 12$.

- *Find the values of f at the endpoints of the interval $[-3,3]$.*

  We have $f(-3) = -4$ and $f(3) = -40$.

- *Compare the values obtained in step 1 and step 2 and return the absolute maximum and the absolute minimum values of f.*

  The absolute maximum value is given by $\max\{-15, 12, -4, -40\} = \mathbf{12}$ and the absolute minimum value is given by $\min\{-15, 12, -4, -40\} = -\mathbf{40}$.