Calculus I - MAC 2311 - Section 007

Quiz 4 - Solutions

10/19/2017

Logarithmic differentiation

You want to differentiate the function $f(x)$ by using logarithmic differentiation:

♦ **Step 0:** *Set* $y = f(x)$.

♦ **Step 1:** *Take the natural logarithm both sides in the equation* $y = f(x)$ *and use the Laws of Logarithms to simplify your right-hand expression.*

♦ **Step 2:** *Differentiate both sides implicitly with respect to* x.

♦ **Step 3:** *Solve your resulting equation for* $\frac{dy}{dx}$ *and, at the end, do not forget that* $y = f(x)$...

1) [5 points] Use logarithmic differentiation to compute the derivative of the following function:

$$f(x) = \frac{\cos^3(x)}{e^{2x} \cdot (x^4 - 2x^2 + 5x)^7}.$$

Solution:

♦ **Step 0:**

$$y = \frac{\cos^3(x)}{e^{2x} \cdot (x^4 - 2x^2 + 5x)^7}.$$

♦ **Step 1:**

$$\begin{aligned} \ln(y) = \ln\left(\frac{\cos^3(x)}{e^{2x} \cdot (x^4 - 2x^2 + 5x)^7}\right) &= \\ &= \ln(\cos^3(x)) - \ln\left(e^{2x} \cdot (x^4 - 2x^2 + 5x)^7\right) = \\ &= \ln(\cos^3(x)) - \left[\ln(e^{2x}) + \ln\left((x^4 - 2x^2 + 5x)^7\right)\right] = \\ &= 3\ln(\cos(x)) - 2x - 7\ln(x^4 - 2x^2 + 5x). \end{aligned}$$

During the simplification we used (in the order) the following facts:

⋆ $\ln\left(\frac{x}{y}\right) = \ln(x) - \ln(y)$,

⋆ $\ln(xy) = \ln(x) + \ln(y)$,

- ⋆ $\ln(x^r) = r\ln(x)$,
- ⋆ $\ln(e^x) = x$.

The resulting equation is

$$\ln(y) = 3\ln(\cos(x)) - 2x - 7\ln(x^4 - 2x^2 + 5x).$$

♦ **Step 2:**

$$\frac{d}{dx}(\ln(y)) = \frac{d}{dx}\left[3\ln(\cos(x)) - 2x - 7\ln(x^4 - 2x^2 + 5x)\right]$$
$$\Downarrow \quad \text{implicit differentiation}$$
$$\frac{1}{y}\frac{dy}{dx} = \frac{d}{dx}(3\ln(\cos(x))) - \frac{d}{dx}(2x) - 7\frac{d}{dx}\left(\ln(x^4 - 2x^2 + 5x)\right)$$
$$\Downarrow$$
$$\frac{1}{y}\frac{dy}{dx} = \frac{-3\sin(x)}{\cos(x)} - 2 - 7\frac{4x^3 + 4x + 5}{x^4 - 2x^2 + 5x}$$

♦ **Step 3:**

$$\frac{1}{y}\frac{dy}{dx} = \frac{-3\sin(x)}{\cos(x)} - 2 - 7\frac{4x^3 + 4x + 5}{x^4 - 2x^2 + 5x}$$
$$\Downarrow$$
$$\frac{dy}{dx} = y\left(\frac{-3\sin(x)}{\cos(x)} - 2 - 7\frac{4x^3 + 4x + 5}{x^4 - 2x^2 + 5x}\right)$$
$$\Downarrow$$
$$\frac{dy}{dx} = \frac{\cos^3(x)}{e^{2x}\cdot(x^4 - 2x^2 + 5x)^7}\left(\frac{-3\sin(x)}{\cos(x)} - 2 - 7\frac{4x^3 + 4x + 5}{x^4 - 2x^2 + 5x}\right)$$

2) [5 points] Compute the derivative of the following function:

$$f(x) = x^{\sin(2x)}.$$

Solution:

⋆ **I method** : Logarithmic differentiation

♦ **Step 0:**

$$y = x^{\sin(2x)}.$$

♦ **Step 1:**

$$\ln(y) = \ln\left(x^{\sin(2x)}\right) = \sin(2x)\ln(x).$$

♦ **Step 2:**

$$\frac{d}{dx}(\ln(y)) = \frac{d}{dx}[\sin(2x)\ln(x)]$$
$$\Downarrow$$
$$\frac{1}{y}\frac{dy}{dx} = 2\cos(2x)\ln(x) + \frac{\sin(2x)}{x}$$

♦ **Step 3:**

$$\begin{aligned}\frac{dy}{dx} &= y\left(2\cos(2x)\ln(x) + \frac{\sin(2x)}{x}\right)\\ &= x^{\sin(2x)}\left(2\cos(2x)\ln(x) + \frac{\sin(2x)}{x}\right).\end{aligned}$$

⋆ **II method**

By using the identity $e^{\ln(x)} = x$, we can rewrite the function in the following way:

$$f(x) = x^{\sin(2x)} = e^{\ln\left(x^{\sin(2x)}\right)} = e^{\sin(2x)\ln(x)}.$$

Hence we have:

$$\begin{aligned}f'(x) &= \left(e^{\sin(2x)\ln(x)}\right)' =\\ &= e^{\sin(2x)\ln(x)}\left(\sin(2x)\ln(x)\right)' =\\ &= e^{\sin(2x)\ln(x)}\left(2\cos(2x)\ln(x) + \frac{\sin(2x)}{x}\right) =\\ &= x^{\sin(2x)}\left(2\cos(2x)\ln(x) + \frac{\sin(2x)}{x}\right).\end{aligned}$$

3) [Bonus] Use logarithmic differentiation to prove the **power rule**.

Solution:

By using logarithmic differentiation, we want to prove that the derivative of the function $f(x) = x^n$ is $f'(x) = nx^{n-1}$.

♦ **Step 0:**

$$y = x^n.$$

♦ **Step 1:**

$$\ln(y) = \ln(x^n) = n\ln(x).$$

♦ **Step 2:**

$$\frac{d}{dx}\left(\ln(y)\right) = \frac{d}{dx}(n\ln(x))$$
$$\Downarrow$$
$$\frac{1}{y}\frac{dy}{dx} = \frac{n}{x}$$

♦ **Step 3:**

$$\frac{dy}{dx} = y\frac{n}{x} = x^n \cdot \frac{n}{x} = nx^{n-1}.$$