Calculus I - MAC 2311

Homework - Review Test 2 - Solutions

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Ex 1. (20 points) Compute the derivatives of the following functions (and show your work):

a) $f(x) = \sqrt{x} + \dfrac{1}{x} + 8 \Rightarrow f'(x) = \dfrac{1}{2\sqrt{x}} - \dfrac{1}{x^2}$.

In more steps:

$$f'(x) = \left[\sqrt{x} + \frac{1}{x} + 8\right]' \overset{\text{sum rule}}{=} [\sqrt{x}]' + \left[\frac{1}{x}\right]' + [8]' = \left[x^{\frac{1}{2}}\right]' + \left[x^{-1}\right]' + [8]' \overset{\text{power rule}}{=}$$
$$= \frac{1}{2}x^{\frac{1}{2}-1} + (-1)x^{-1-1} + 0 = \frac{1}{2}x^{-\frac{1}{2}} - x^{-2} = \frac{1}{2\sqrt{x}} - \frac{1}{x^2}.$$

b) $f(x) = \cos(x^8) \Rightarrow f'(x) \overset{\text{chain rule}}{=} -\sin(x^8) \cdot \left[x^8\right]' = -\sin(x^8) \cdot 8x^7 = -8x^7 \sin(x^8)$.

c) $f(x) = \cos^8(x) = (\cos(x))^8 \Rightarrow f'(x) \overset{\text{chain rule}}{=} 8(\cos(x))^7 \cdot [\cos(x)]' =$
$= 8(\cos(x))^7 \cdot (-\sin(x)) = -8\cos^7(x)\sin(x)$.

d) $f(t) = \sqrt{t^5} = (t^5)^{\frac{1}{2}} = t^{\frac{5}{2}} \Rightarrow f'(t) \overset{\text{power rule}}{=} \frac{5}{2}t^{\frac{5}{2}-1} = \frac{5}{2}t^{\frac{3}{2}} = \frac{5}{2}(t^3)^{\frac{1}{2}} = \frac{5}{2}\sqrt{t^3}$.

e) $f(x) = \dfrac{1}{\sqrt{\pi}} \Rightarrow f'(x) = 0$ (the derivative of a constant is zero).

f) $f(x) = x^2\ln(x) \Rightarrow f'(x) \overset{\text{product rule}}{=} \left[x^2\right]'\ln(x) + x^2 \cdot [\ln(x)]' = 2x\ln(x) + x^2 \cdot \dfrac{1}{x} =$
$= 2x\ln(x) + x$.

g) $f(x) = \dfrac{e^x}{\sin(3x)} \Rightarrow f'(x) \overset{\text{quotient rule}}{=} \dfrac{[e^x]'\sin(3x) - e^x\,[\sin(3x)]'}{(\sin(3x))^2} \overset{\text{chain rule}}{=}$
$$= \frac{e^x\sin(3x) - e^x \cdot \cos(3x) \cdot [3x]'}{\sin^2(3x)} = \frac{e^x\sin(3x) - e^x \cdot \cos(3x) \cdot 3}{\sin^2(3x)} = \frac{e^x(\sin(3x) - 3\cos(3x))}{\sin^2(3x)}.$$

h) $f(x) = e^{\ln(\sin(x))} = \sin(x) \Rightarrow f'(x) = \cos(x)$.

Clever way:

$f(x) = e^{\ln(\sin(x))} = \sin(x) \Rightarrow f'(x) = \cos(x)$.
Here we used the fact that $e^{\ln x} = x$.

Less clever (but also accepted) way:

$$f(x) = e^{\ln(\sin(x))} = \sin(x) \Rightarrow f'(x) \overset{\text{chain rule}}{=} e^{\ln(\sin(x))} \cdot [\ln(\sin(x))]' \overset{\text{chain rule}}{=}$$
$$= e^{\ln(\sin(x))} \cdot \frac{1}{\sin(x)} \cdot [\sin(x)]' = e^{\ln(\sin(x))} \cdot \frac{1}{\sin(x)} \cdot \cos(x) = e^{\ln(\sin(x))} \cdot \frac{\cos(x)}{\sin(x)}$$

You can actually stop here, but using again the identity $e^{\ln x} = x$ you can recover the previous solution:

$$e^{\ln(\sin(x))} \cdot \frac{\cos(x)}{\sin(x)} = \sin(x) \cdot \frac{\cos(x)}{\sin(x)} = \cos(x).$$

i) $f(x) = \sin(\tan(8x)) \Rightarrow f'(x) \overset{\text{chain rule}}{=} \cos(\tan(8x)) \cdot [\tan(8x)]' \overset{\text{chain rule}}{=}$
$= \cos(\tan(8x)) \cdot \dfrac{1}{\cos^2(8x)} \cdot [8x]' = \cos(\tan(8x)) \cdot \dfrac{1}{\cos^2(8x)} \cdot 8 = \dfrac{8\cos(\tan(8x))}{\cos^2(8x)}.$

j) $f(u) = e^u \cos(u) \tan(u)$

Clever way:

$f(u) = e^u \cos(u) \tan(u) = e^u \cos(u) \dfrac{\sin(u)}{\cos(u)} = e^u \sin(u) \Rightarrow$

$\Rightarrow f'(u) \overset{\text{product rule}}{=} e^u \sin(u) + e^u \cos(u) = e^u(\sin(u) + \cos(u)).$

Less clever (but also accepted) way:

$f(u) = e^u \cos(u) \tan(u) \Rightarrow f'(u) \overset{\text{product rule}}{=} [e^u \cos(u)]' \cdot \tan(u) + e^u \cos(u) \cdot [\tan(u)]' \overset{\text{product rule}}{=}$

$= ([e^u]' \cos(u) + e^u [\cos(u)]') \tan(u) + e^u \cos(u) \cdot \dfrac{1}{\cos^2(u)} =$

$= (e^u \cos(u) + e^u(-\sin(u))) \tan(u) + e^u \cdot \dfrac{1}{\cos(u)} =$

$= e^u \cos(u) \tan(u) + e^u(-\sin(u)) \tan(u) + e^u \cdot \dfrac{1}{\cos(u)} =$

$= e^u \left(\cos(u) \tan(u) - \sin(u) \tan(u) + \dfrac{1}{\cos(u)} \right).$

It is fine if you stop here, but using again the identity $\tan(u) = \frac{\sin(u)}{\cos(u)}$ you can recover the previous solution:

$e^u \left(\cos(u) \tan(u) - \sin(u) \tan(u) + \dfrac{1}{\cos(u)} \right) = e^u \left(\cos(u) \dfrac{\sin(u)}{\cos(u)} - \sin(u) \dfrac{\sin(u)}{\cos(u)} + \dfrac{1}{\cos(u)} \right) =$

$= e^u \left(\dfrac{\sin(u)\cos(u) - \sin^2(u) + 1}{\cos(u)} \right) \overset{\sin^2(u)+\cos^2(u)=1}{=} e^u \left(\dfrac{\sin(u)\cos(u) + \cos^2(u)}{\cos(u)} \right) =$

$= e^u (\sin(u) + \cos(u)).$

Ex 2. (10+10 points) Consider the curve given by the equation

$$x^2y^2 + xy = 2.$$

a) Use implicit differentiation to find y' $\left(\text{i.e. } \dfrac{dy}{dx} \right)$.

b) Find an equation of the tangent line to the above curve at the point $(1, 1)$.

Solution:

a) If in the equation

$$x^2y^2 + xy = 2 \tag{1}$$

we choose x as the independent variable, we say that y is defined *implicitly* in function of x. We can highlight this fact by rewriting the equation (1) in the following way:

$$x^2 \cdot (y(x))^2 + x \cdot y(x) = 2$$

Hence, we may find the derivative $\frac{dy}{dx}$ by using implicit differentiation (we recall that in the Leibniz notation $\frac{dy}{dx}$ the variable y represents the *dependent variable* and x the *independent variable*).
We take the derivative of each side of equation (1) with respect to x (remembering to treat y as a function of x), and apply the rules of differentiation:

$$\frac{d}{dx}\left(x^2y^2+xy\right)=\frac{d}{dx}(2)$$

$\Downarrow$ sum rule

$$\frac{d}{dx}\left(x^2y^2\right)+\frac{d}{dx}(xy)=0$$

$\Downarrow$ product rule

$$\frac{d}{dx}\left(x^2\right)\cdot y^2+x^2\cdot\frac{d}{dx}\left(y^2\right)+\frac{d}{dx}(x)\cdot y+x\cdot\frac{d}{dx}(y)=0$$

$\Downarrow$ chain rule

$$2xy^2+x^2\cdot\frac{d}{dy}\left(y^2\right)\cdot\frac{dy}{dx}+y+x\cdot\frac{dy}{dx}=0$$

$\Downarrow$

$$2xy^2+x^2\cdot 2y\cdot\frac{dy}{dx}+y+x\cdot\frac{dy}{dx}=0$$

Now we have an ordinary linear equation where the unknown we want to solve for is $\dfrac{dy}{dx}$. From the last step we obtain:

$$(2x^2y+x)\cdot\frac{dy}{dx}=-2xy^2-y,$$

which implies

$$\frac{dy}{dx}=\frac{-2xy^2-y}{2x^2y+x}. \tag{2}$$

b) If $P(x,y)$ is a point on the curve $\mathcal{C}$ described by the equation

$$x^2y^2+xy=2,$$

i.e. the coordinates x and y of P make the previous equation true, we have that the slope of the tangent line to the curve $\mathcal{C}$ at $P(x,y)$ is given by:

$$\frac{dy}{dx}=\frac{-2xy^2-y}{2x^2y+x}.$$

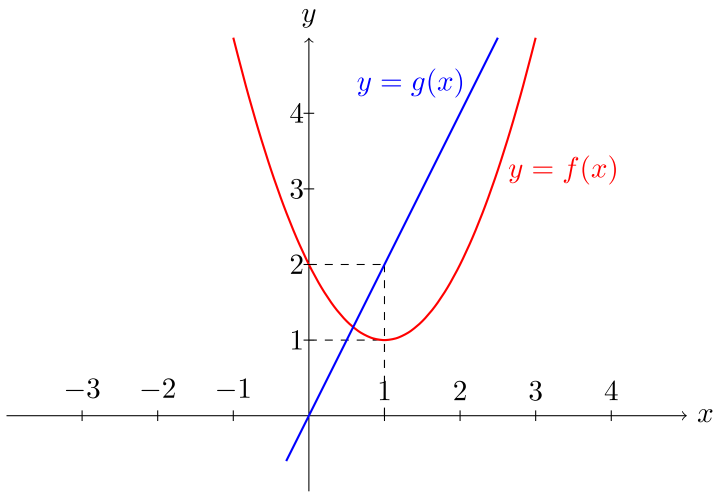
Hence, for the point $(1,1)$, by substituting $x=1$ and $y=1$ in the previous formula, we have:

$$\frac{dy}{dx}=\frac{-2-1}{2+1}=-1.$$

We deduce that the equation of the tangent line to the curve $\mathcal{C}$ at the point $(1,1)$ is $y-1=-1\cdot(x-1)$, i.e.

$$y=-x+2.$$

Ex 3. (5+5+5+5 points)



Let f and g be the functions whose graphs are shown above and let

$$h(x) = f(x) + g(x), \qquad u(x) = f(x)g(x), \qquad v(x) = \frac{f(x)}{g(x)}, \qquad w(x) = g(f(x)).$$

Compute $h'(1)$, $u'(1)$, $v'(1)$ and $w'(1)$.

Solution:

By using the differentiation rules (respectively sum, product, quotient and chain rule) we have:

$$\begin{aligned} h'(x) &= f'(x) + g'(x); \\ u'(x) &= f'(x)g(x) + f(x)g'(x); \\ v'(x) &= \frac{f'(x)g(x) - f(x)g'(x)}{(g(x))^2}; \\ w'(x) &= g'(f(x))f'(x). \end{aligned}$$

Hence, in order to compute $h'(1)$, $u'(1)$, $v'(1)$ and $w'(1)$, we need to find before the values for $f(1)$, $g(1)$, $f'(1)$, $g'(1)$.

Easily from the graphs of f and g we get $f(1) = 1$ and $g(1) = 2$.

For computing $f'(1)$ (respectively $g'(1)$) we need to find the slope of the tangent line to the graph $y = f(x)$ (respectively $y = g(x)$) at the point $(1, f(1))$ (respectively $(1, g(1))$).

In the first case, the tangent line is parallel to the x-axis, so that its slope is equal to 0. This means that

$$f'(1) = 0.$$

In the second case, the graph $y = g(x)$ is a line, which coincides with the tangent line to itself at each of its points. Thus, we can compute its slope by using the coordinates of two of its points, for example $(2, 1)$ and $(0, 0)$, and we have:

$$g'(1) = \frac{2-0}{1-0} = 2.$$

We are now ready for computing $h'(1)$, $u'(1)$, $v'(1)$ and $w'(1)$:

$$\begin{aligned}
h'(1) &= f'(1) + g'(1) = 0 + 2 = 2;\\
u'(1) &= f'(1)g(1) + f(1)g'(1) = 0 \cdot 2 + 1 \cdot 2 = 2;\\
v'(1) &= \frac{f'(1)g(1) - f(1)g'(1)}{(g(1))^2} = \frac{0 \cdot 2 - 1 \cdot 2}{2^2} = -\frac{1}{2};\\
w'(1) &= g'(f(1))f'(1) = g'(f(1)) \cdot 0 = 0.
\end{aligned}$$

Ex 4. (5+5+10 points) A couple of alligators meets at the intersection of Bruce B. Downs Blvd and Fowler Ave for organizing a romantic dinner. The male alligator starts running east at a speed of 0.4 miles per minute to chase a USF student. At the same time the female alligator starts running north at a speed of 0.3 miles per minute to chase a USF instructor.

At a given time t (measured in minutes), let $x(t)$ be the distance between the male alligator and the intersection point, $y(t)$ be the distance between the female alligator and the intersection point and $z(t)$ be the distance between the two alligators.

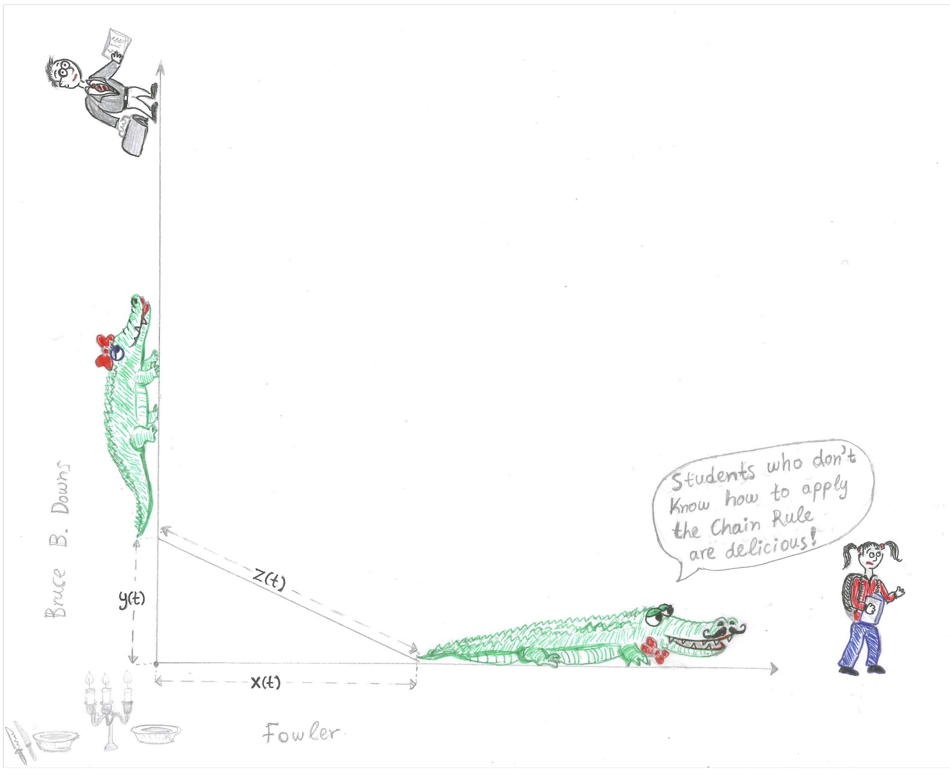
a) Find an equation that relates $x(t), y(t)$ and $z(t)$.

b) Compute $x(5)$, $y(5)$ and $z(5)$.

c) At what rate is the distance between the two alligators increasing after 5 minutes?

Solution:

First, let us understand the problem, by drawing a picture and finding and naming the quantities which are related.



At a given time t:

$\mathbf{x} = \mathbf{x}(\mathbf{t})$: the distance between the male alligator and the intersection point
$\mathbf{y} = \mathbf{y}(\mathbf{t})$: the distance between the female alligator and the intersection point
$\mathbf{z} = \mathbf{z}(\mathbf{t})$: the distance between the two alligators

It is also a good idea to write what we know and what we wish to find:

Known: $\dfrac{dx}{dt} = 0.4$ and $\dfrac{dy}{dt} = 0.3$

Want to find: $\dfrac{dz}{dt} = ?$ when $t = 5$ minutes.

a) By Pythagoras Theorem the quantities $x(t)$, $y(t)$ and $z(t)$ are related by the following equation: $z^2 = x^2 + y^2$; i.e. for all t:

$$(z(t))^2 = (x(t))^2 + (y(t))^2. \tag{3}$$

b) Since the alligators are moving at a constant velocity (0.4 miles/minute in the case of the male alligator and 0.3 miles/minutes in the case of the female alligator) we have:

$$x(t) = 0.4t \qquad \text{and} \qquad y(t) = 0.3t$$

Hence

$$x(5) = 0.4 \cdot 5 = 2 \text{ miles} \qquad \text{and} \qquad y(5) = 0.3 \cdot 5 = 1.5 \text{ miles}.$$

For finding $z(5)$ we use the equation (3) for $t = 5$:

$$z(5) = \sqrt{(x(5))^2 + (y(5))^2} = \sqrt{2^2 + 1.5^2} = \sqrt{4 + 2.25} = \sqrt{6.25} = 2.5 \text{ miles}$$

c) Equation (3) shows how the quantities are related at each time t. We are interested in how the corresponding rates relate. For that, we differentiate both sides of equation (3) with respect to t:

$$\frac{d}{dt}(z(t))^2 = \frac{d}{dt}(x(t))^2 + \frac{d}{dt}(y(t))^2 \quad \overset{\text{chain rule}}{\Leftrightarrow} \quad 2z(t)\frac{dz}{dt} = 2x(t)\frac{dx}{dt} + 2y(t)\frac{dy}{dt}$$

By isolating $\dfrac{dz}{dt}$ in the last equation we get:

$$\frac{dz}{dt} = \frac{2x(t)\dfrac{dx}{dt} + 2y(t)\dfrac{dy}{dt}}{2z(t)} \tag{4}$$

We replace in equation (4) all the known information we collected in the previous steps and compute it for $t = 5$. We obtain:

$$\frac{dz}{dt} = \frac{2x(5) \cdot 0.4 + 2y(5) \cdot 0.3}{2z(5)} = \frac{2 \cdot 2 \cdot 0.4 + 2 \cdot 1.5 \cdot 0.3}{2 \cdot 2.5} = 0.5 \text{ miles/minute}.$$

We conclude that after 5 minutes the alligators are moving away at a rate of 0.5 miles/minute (*Don't forget to put the units in the end!*).

Ex 5. (5+5+5+5 points) Which statements are True/False? Justify your answers.

a) If $f(0) = g(0)$ then $f'(0) = g'(0)$.

False. In order to show that the statement is false, it is enough to provide an example of two functions $f(x)$ and $g(x)$ such that $f(0) = g(0)$ and $f'(0) \neq g'(0)$. Let $f(x) = x$ and $g(x) = x^2$. Then $f'(x) = 1$ and $g'(x) = 2x$. Hence we have $f(0) = g(0) = 0$ but $f'(0) = 1$ and $g'(0) = 0$.

b) If $f(x) = \cos(x)$ then $f''(0) = 0$.

False. We have $f'(x) = -\sin(x)$ and $f''(x) = (f'(x))' = (-\sin(x))' = -\cos(x)$ so that $f''(0) = -\cos(0) = -1$.

c) If the graphs of two functions f and g have the same tangent line at 0 then $f'(0) = g'(0)$.

True. Indeed $f'(0)$ (resp. $g'(0)$) represents the slope of the tangent line at 0 to the curve $y = f(x)$ (resp. $y = g(x)$).

d) The function $f(x) = |x - 2|$ is differentiable at 2 since it is continuous at 2.

False. The function $f(x) = |x - 2|$ is not differentiable at 2, even if it is continuos at 2 (in class we saw the theorem *differentiable at a* $\Rightarrow$ *continuous at a* but the converse is in general not true).

We show that by proving that

$$\lim_{h\to 0} \frac{f(2+h) - f(2)}{h}$$

does not exist. Indeed we have:

$$\lim_{h\to 0^-} \frac{f(2+h) - f(2)}{h} = \lim_{h\to 0^-} \frac{|2+h-2| - |2-2|}{h} = \lim_{h\to 0^-} \frac{|h| - 0}{h} \overset{h<0}{=} \lim_{h\to 0^-} \frac{-h}{h} = -1$$

and

$$\lim_{h\to 0^+} \frac{f(2+h) - f(2)}{h} = \lim_{h\to 0^+} \frac{|2+h-2| - |2-2|}{h} = \lim_{h\to 0^+} \frac{|h| - 0}{h} \overset{h>0}{=} \lim_{h\to 0^+} \frac{h}{h} = 1.$$

Since the left-hand and the right-hand limits are not equal, then $\lim_{h\to 0} \frac{f(2+h)-f(2)}{h}$, and consequently $f'(2)$, do not exist.