Calculus I - MAC 2311 - Section 007

Homework - Review Test 1 - Solutions

Annamaria Iezzi

1) Compute the following limits (and show all your work):

a) $\lim_{x\to 0} \frac{x}{x^2+1} = \frac{0}{0^2+1} = \frac{0}{1} = \mathbf{0}$

b) $\lim_{x\to -1} \frac{x+1}{x^2+3x+2} = \lim_{x\to -1} \frac{x+1}{(x+1)(x+2)} = \lim_{x\to -1} \frac{1}{x+2} = \frac{1}{-1+2} = \mathbf{1}$

c) $\lim_{x\to 1} \frac{x^3-x^2+x-1}{x-1} = \lim_{x\to 1} \frac{x^2(x-1)+(x-1)}{x-1} = \lim_{x\to 1} \frac{(x^2+1)(x-1)}{x-1} = \lim_{x\to 1} x^2+1 = \mathbf{2}$

d) $\lim_{x\to 4} \frac{-\sqrt{x}+2}{x-4} = \lim_{x\to 4} \frac{-\sqrt{x}+2}{x-4} \cdot \frac{-\sqrt{x}-2}{-\sqrt{x}-2} = \lim_{x\to 4} \frac{(-\sqrt{x})^2-2^2}{(x-4)(-\sqrt{x}-2)} = \lim_{x\to 4} \frac{x-4}{(x-4)(-\sqrt{x}-2)} =$
$= \lim_{x\to 4} \frac{1}{-\sqrt{x}-2} = \frac{1}{-\sqrt{4}-2} = \frac{1}{-2-2} = \mathbf{-\frac{1}{4}}$

e) $\lim_{x\to 0} \frac{x}{\sqrt{2+x}-\sqrt{2-x}} = \lim_{x\to 0} \frac{x}{\sqrt{2+x}-\sqrt{2-x}} \cdot \frac{\sqrt{2+x}+\sqrt{2-x}}{\sqrt{2+x}+\sqrt{2-x}} =$
$= \lim_{x\to 0} \frac{x(\sqrt{2+x}+\sqrt{2-x})}{(\sqrt{2+x})^2-(\sqrt{2-x})^2} = \lim_{x\to 0} \frac{x(\sqrt{2+x}+\sqrt{2-x})}{2+x-(2-x)} = \lim_{x\to 0} \frac{x(\sqrt{2+x}+\sqrt{2-x})}{2x}$
$= \lim_{x\to 0} \frac{\sqrt{2+x}+\sqrt{2-x})}{2} = \frac{\sqrt{2}+\sqrt{2}}{2} = \mathbf{\sqrt{2}}$

f) $\lim_{x\to \infty} \frac{2x^5-x^3+3}{6x^5+1} = \lim_{x\to \infty} \frac{x^5\left(2-\frac{1}{x^2}+\frac{3}{x^5}\right)}{x^5\left(6+\frac{1}{x^5}\right)} = \lim_{x\to \infty} \frac{2-\frac{1}{x^2}+\frac{3}{x^5}}{6+\frac{1}{x^5}} = “\frac{2-\frac{1}{\infty}+\frac{3}{\infty}}{6+\frac{1}{\infty}}” =$
$= \frac{2-0+0}{6+0} = \mathbf{\frac{1}{3}}$

g) $\lim_{x\to -\infty} \frac{x^3-x^2+x-1}{x-1} = \lim_{x\to -\infty} \frac{x^3\left(1-\frac{1}{x}+\frac{1}{x^2}-\frac{1}{x^3}\right)}{x\left(1-\frac{1}{x}\right)} = \lim_{x\to -\infty} \frac{x^2\left(1-\frac{1}{x}+\frac{1}{x^2}-\frac{1}{x^3}\right)}{1-\frac{1}{x}} =$
$= “\frac{(-\infty)^2\left(1-\frac{1}{-\infty}+\frac{1}{\infty}-\frac{1}{-\infty}\right)}{1-\frac{1}{-\infty}}” = “\frac{\infty\cdot 1}{1}” = \infty$

h) $\lim_{t\to \infty} \frac{t+1}{t^2+1} = \lim_{t\to \infty} \frac{t\left(1+\frac{1}{t}\right)}{t^2\left(1+\frac{1}{t^2}\right)} = \lim_{t\to \infty} \frac{1+\frac{1}{t}}{t\left(1+\frac{1}{t^2}\right)} = “\frac{1+\frac{1}{\infty}}{\infty\cdot\left(1+\frac{1}{\infty}\right)}” = “\frac{1}{\infty\cdot 1}” = “\frac{1}{\infty}” = \mathbf{0}$

i) $\lim_{x\to-\infty}(x+\sqrt{3-x}) = \lim_{x\to-\infty} x\left(1+\frac{\sqrt{3-x}}{x}\right) = \lim_{x\to-\infty} x\left(1+\frac{\sqrt{3-x}}{-\sqrt{x^2}}\right) =$
$= \lim_{x\to-\infty} x\left(1-\sqrt{\frac{3-x}{x^2}}\right) = \lim_{x\to-\infty} x\left(1-\sqrt{\frac{3}{x^2}-\frac{1}{x}}\right) = \text{``}-\infty\cdot\left(1-\sqrt{\frac{3}{\infty}-\frac{1}{-\infty}}\right)\text{''} =$
$= \text{``}-\infty\cdot\left(1-\sqrt{0-0}\right)\text{''} = \text{``}-\infty\cdot 1\text{''} = -\infty$

Here, between the second and the third step we used the fact that when $x<0$ (here $x\to-\infty$) then $x=-|x|=-\sqrt{x^2}$, and between the third and the fourth we used the fact that $\frac{\sqrt{a}}{\sqrt{b}}=\sqrt{\frac{a}{b}}$.

j) $\lim_{x\to2}\frac{x-3}{(x-2)^2}$

We will solve this limit by computing the left-hand and the right-hand limits:

$\lim_{x\to2^-}\frac{x-3}{(x-2)^2} = \text{``}\frac{2-3}{(0^-)^2}\text{''} = \text{``}\frac{-1}{0^+}\text{''} = -\infty$. (Remember that when $x\to2^-$ then $x<2$ so that $x-2<0$ and $x-2\to0^-$).

$\lim_{x\to2^+}\frac{x-3}{(x-2)^2} = \text{``}\frac{2-3}{(0^+)^2}\text{''} = \text{``}\frac{-1}{0^+}\text{''} = -\infty$.

Since $\lim_{x\to2^-}\frac{x-3}{(x-2)^2} = \lim_{x\to2^+}\frac{x-3}{(x-2)^2} = -\infty$ then $\lim_{x\to2}\frac{x-3}{(x-2)^2} = -\infty$

k) $\lim_{x\to0}\frac{x^3-2}{x}$

We will solve this limit by computing the left-hand and the right-hand limits:

$\lim_{x\to0^-}\frac{x^3-2}{x} = \text{``}\frac{0-2}{0^-}\text{''} = \text{``}\frac{-2}{0^-}\text{''} = \text{``}-2\cdot\frac{1}{0^-}\text{''} = \text{``}-2\cdot(-\infty)\text{''} = \infty$.

$\lim_{x\to0^+}\frac{x^3-2}{x} = \text{``}\frac{0-2}{0^+}\text{''} = \text{``}\frac{-2}{0^+}\text{''} = \text{``}-2\cdot\frac{1}{0^+}\text{''} = \text{``}-2\cdot\infty\text{''} = -\infty$.

Since $\lim_{x\to0^-}\frac{x^3-2}{x} \neq \lim_{x\to0^+}\frac{x^3-2}{x}$ then $\lim_{x\to0}\frac{x^3-2}{x}$ **does not exist**.

l) $\lim_{\alpha\to0}\frac{\sin(3\alpha)}{6\alpha} = \lim_{\alpha\to0}\frac{1}{2}\cdot\frac{\sin(3\alpha)}{3\alpha} = \frac{1}{2}\cdot\lim_{\alpha\to0}\frac{\sin(3\alpha)}{3\alpha} = \frac{1}{2}\cdot1 = \mathbf{\frac{1}{2}}$.

m) $\lim_{x\to\frac{\pi}{2}}\frac{\sin(x-\frac{\pi}{2})}{x-\frac{\pi}{2}}$

If we set $\alpha = x-\frac{\pi}{2}$ then when $x\to\frac{\pi}{2}$ we have $\alpha\to0$ and by substitution in the limit we get:

$\lim_{x\to\frac{\pi}{2}}\frac{\sin(x-\frac{\pi}{2})}{x-\frac{\pi}{2}} = \lim_{\alpha\to0}\frac{\sin(\alpha)}{\alpha} = \mathbf{1}$

n) $\lim_{x\to1^-}\frac{-|x-1|}{x-1}$

When $x \to 1^-$ in particular $x < 1$ (or equivalently $x - 1 < 0$) so that $|x-1| = -(x-1)$. Hence we have:

$$\lim_{x\to 1^-} \frac{-|x-1|}{x-1} = \lim_{x\to 1^-} \frac{-(-(x-1))}{x-1} = \lim_{x\to 1^-} \frac{x-1}{x-1} = \mathbf{1}$$

2) Sketch the graph of a function f which is defined for all real numbers and satisfies simultaneously the following:

a) $\lim_{x\to\infty} f(x) = 4$

b) The line $y = -1$ is a horizontal asymptote.

c) $f(0) = 1$.

d) The line $x = 2$ is a vertical asymptote.

e) $\lim_{x\to 2^+} f(x) = -\infty$.

f) $x = 1$ is a solution for the equation $f(x) = 0$.

Solution:

Let us translate some of these conditions geometrically.

a) $\lim_{x\to\infty} f(x) = 4$: this means that the line $y = 4$ is an horizontal asymptote for the graph of the function f.

b) The line $y = -1$ is an horizontal asymptote: this means that $\lim_{x\to\infty} f(x) = -1$ or $\lim_{x\to-\infty} f(x) = -1$. Since we know already from a) that $\lim_{x\to\infty} f(x) = 4$ then we get $\lim_{x\to-\infty} f(x) = -1$

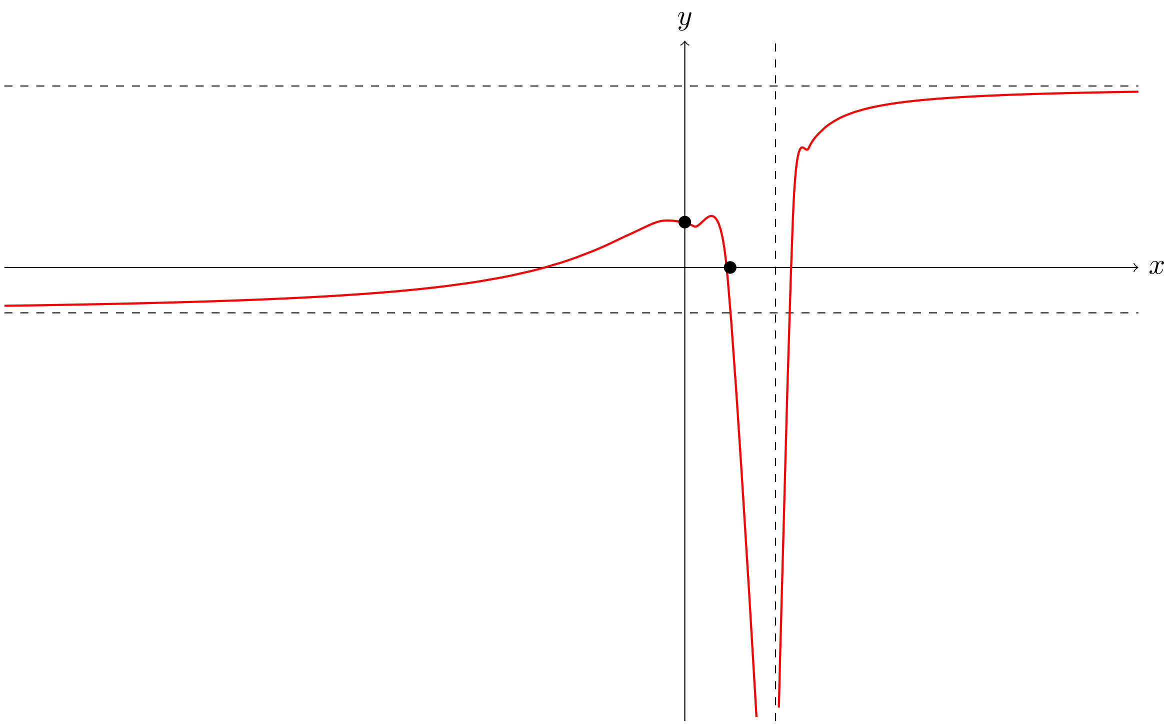
c) $f(0) = 1$: the graph of the function passes through the point $(0, 1)$.

d) The line $x = 2$ is a vertical asymptote.

e) $\lim_{x\to 2^+} f(x) = -\infty$.

f) $x = 1$ is a solution for the equation $f(x) = 0$: this means that $f(1) = 0$ that is the graph of the function passes through the point $(1, 0)$.

Of course there is not an unique function that satisfies simultaneously all these conditions. An example is given by the function whose graph is the following:



3) Let f be the function:

$$f(x) = \begin{cases} \frac{x}{x+1}, & x < -1; \\ x^2 + 2, & -1 \leq x \leq 2; \\ \cos(\pi x) + 5, & x > 2 \end{cases}$$

a) Compute $f(-1)$, $\lim_{x \to (-1)^-} f(x)$, $\lim_{x \to (-1)^+} f(x)$, $f(2)$, $\lim_{x \to 2^-} f(x)$, $\lim_{x \to 2^+} f(x)$.

Solution:

We remark that $f(x)$ is a piecewise function whose branches are respectively defined on the intervals $(-\infty, -1)$, $[-1, 2]$ and $(2, \infty)$.

- When $x = -1$ then $f(x) = x^2 + 2$, hence $f(-1) = (-1)^2 + 2 = 1 + 2 = 3$.
- When $x < -1$ then $f(x) = \frac{x}{x+1}$, hence $\lim_{x \to (-1)^-} f(x) = \lim_{x \to (-1)^-} \frac{x}{x+1} =$ "$\frac{-1}{0^-}$" $= \infty$.
- When $x > -1$ then $f(x) = x^2 + 2$, hence $\lim_{x \to (-1)^+} f(x) = \lim_{x \to (-1)^+} x^2 + 2 = (-1)^2 + 2 = 3$.
- When $x = 2$ then $f(x) = x^2 + 2$, hence $f(2) = (2)^2 + 2 = 6$.
- When $x < 2$ then $f(x) = x^2 + 2$, hence $\lim_{x \to 2^-} f(x) = \lim_{x \to 2^-} x^2 + 2 = (2)^2 + 2 = 6$.

- When $x > 2$ then $f(x) = \cos(\pi x) + 5$, hence $\lim_{x\to 2^+} f(x) = \lim_{x\to 2^+} \cos(\pi x) + 5 = \cos(2\pi) + 5 = 1 + 5 = 6$.

b) Is the function f continuous at $x = -1$? And at $x = 2$?

Solution:

- Since $\lim_{x\to(-1)^-} f(x) = \infty$ the function f is not continuous at $x = 1$ and $x = 1$ is an infinite discontinuity.
- Since $\lim_{x\to 2^-} f(x) = \lim_{x\to 2^+} f(x) = f(2) = 6$ then the function f is continuous at $x = 2$.

4) State the Intermediate Value Theorem. Then, use it to prove that the equation:

$$x^2 + \sin\left(\frac{\pi}{2}x\right) + 2 = 3$$

has at least one solution in $[0, 1]$.

Solution:

Theorem *(Intermediate Value Theorem). Let f be a continuous function on an interval $[a, b]$, with $f(a) \neq f(b)$. Then for every number N between $f(a)$ and $f(b)$ there exists $c \in (a, b)$ such that $f(c) = N$.*

Let

$$f(x) = x^2 + \sin\left(\frac{\pi}{2}x\right) + 2.$$

The function f is a continuous function at all the real numbers, since it is the sum of continuous functions (polynomial function, sinus function, constant function). In particular it is continuous on the interval $[0, 1]$.

We have

$$f(0) = 0 + \sin(0) + 2 = 2 \qquad \text{and} \qquad f(1) = 1 + \sin\left(\frac{\pi}{2}\right) + 2 = 1 + 1 + 2 = 4.$$

By the Intermediate Value Theorem, for all $2 \leq N \leq 4$ there exists a number $c \in (0, 1)$ such that $f(c) = N$. In particular this is true for $N = 3$. Hence the equation $f(x) = 3$ has a solution in $[0, 1]$.

5) Write the equations of the vertical and horizontal asymptotes of the following function:

$$f(x) = \frac{3x^3 + 4x}{x^3 - 2x}.$$

Solution:

We recall that a function has an horizontal asymptote if and only if at least one of the following limits is finite: $\lim_{x\to\infty} f(x)$, $\lim_{x\to-\infty} f(x)$. If $\lim_{x\to\infty} f(x) = L < \infty$ or $\lim_{x\to-\infty} f(x) = L < \infty$ then the line $y = L$ is an horizontal asymptote (it is clear that a function can have at most two different horizontal asymptotes).

In our case we have:

$$\lim_{x\to\infty} \frac{3x^3+4x}{x^3-2x} = \lim_{x\to\infty} \frac{x^3\left(3+\frac{4}{x^2}\right)}{x^3\left(1-\frac{2}{x^2}\right)} = \lim_{x\to\infty} \frac{3+\frac{4}{x^2}}{1-\frac{2}{x^2}} = 3.$$

In a totally analogous way we can show that $\lim_{x\to-\infty} \frac{3x^3+4x}{x^3-2x} = 3$.

We deduce that the line $y = 3$ is the unique horizontal asymptote for the function f.

We recall that a function has vertical asymptotes in correspondence of all the points that are infinite discontinuities. If the point $x = a$ is an infinite discontinuity, then the line $x = a$ is a vertical asymptote.

In the case of a rational function the infinite discontinuities have to be found among the values of x that make the denominator equal to 0 (but possibly some of these values are not infinite discontinuities...).

Let us consider now our function $f(x) = \frac{3x^3+4x}{x^3-2x}$. We can factor its denominator as $x^3 - 2x = x(x^2-2) = x(x-\sqrt{2})(x+\sqrt{2})$. Let us check if $x = 0, x = \sqrt{2}$ and $x = -\sqrt{2}$ are infinite discontinuities.

- $\lim_{x\to 0} f(x) = \lim_{x\to 0} \frac{3x^3+4x}{x^3-2x} = \lim_{x\to 0} \frac{x(3x^2+4)}{x(x^2-2)} = \lim_{x\to 0} \frac{3x^2+4}{x^2-2} = \frac{0+4}{0-2} = -2.$
- $\lim_{x\to\sqrt{2}^-} f(x) = \lim_{x\to\sqrt{2}^-} \frac{3x^3+4x}{x^3-2x} = \lim_{x\to\sqrt{2}^-} \frac{3x^3+4x}{x(x-\sqrt{2})(x+\sqrt{2})} = \text{“}\frac{3(\sqrt{2})^3+4(\sqrt{2})}{\sqrt{2}(0^-)(2\sqrt{2})}\text{”} =$
  $= \text{“a positive guy} \cdot \frac{1}{0^-}\text{”} = -\infty.$
- $\lim_{x\to(-\sqrt{2})^-} f(x) = \lim_{x\to(-\sqrt{2})^-} \frac{3x^3+4x}{x^3-2x} = \lim_{x\to(-\sqrt{2})^-} \frac{3x^3+4x}{x(x-\sqrt{2})(x+\sqrt{2})} = \text{“}\frac{3(-\sqrt{2})^3+4(-\sqrt{2})}{-\sqrt{2}(-2\sqrt{2})(0^-)}\text{”} =$
  $= \text{“a negative guy} \cdot \frac{1}{0^-}\text{”} = \infty.$

We obtain that only $x = \sqrt{2}$ and $x = -\sqrt{2}$ are vertical asymptotes and they are all the vertical asymptotes of the function f.

6) Find the derivative (or the instantaneous rate of change) of the function $f(x) = \sqrt{x}+1$ at the point $a = 4$. Then, write the equation of the tangent line to the curve $y = f(x)$ at the point $P(4, 3)$.

Solution:

By definition we have that the derivative of a function f at a point a is given by:

$$f'(a) = \lim_{h\to 0} \frac{f(a+h) - f(a)}{h}.$$

If this limite exists and is finite, then the equation of the tangent line to the curve $y = f(x)$ at the point $(a, f(a))$ is given by:

$$y - f(a) = f'(a)(x - a).$$

In our exercice $f(x) = \sqrt{x} + 1$ and $a = 4$. Then:

$$f'(4) = \lim_{h\to 0} \frac{f(4+h) - f(4)}{h} = \lim_{h\to 0} \frac{\sqrt{4+h} + 1 - (\sqrt{4} + 1)}{h} = \lim_{h\to 0} \frac{\sqrt{4+h} - 2}{h} \cdot \frac{\sqrt{4+h} + 2}{\sqrt{4+h} + 2} =$$
$$= \lim_{h\to 0} \frac{4+h-4}{h(\sqrt{4+h} + 2)} = \lim_{h\to 0} \frac{1}{\sqrt{4+h} + 2} = \frac{1}{4}.$$

The equation of the tangent line to the curve $y = f(x)$ at the point $P(4, f(4)) = (4, 3)$ is given by $y - 3 = \frac{1}{4}(x - 4)$, that is $y = \frac{1}{4}x + 2.$