Bridge - MGF 3301 - Section 001

TEST 3

04/22/2020

Instructions

This test contains 6 exercises. The total number of points is 120, but your grade will be the minimum between your score and 110 (you can get up to 10 bonus points).

In each proof you may use a result proved in class, in the homework or in previous exercises of this test. In this case just state clearly which result you are using, but there is no need of proving it again.

Neatness and clarity are important. You will lose credit if we cannot decipher your answer.

This exam is open-book and open-notes. You are not allow to discuss the exercises of this test with any other student.

When you have completed your work, please submit it by 11am on Gradescope.com, under the assignment *Test 3*. Remember that you have to submit one unique pdf.

For that as always you will have three options:

a) If you have a **tablet with a stylus**, write your answers to the exercises directly on this pdf, in the provided blank spaces. When you have completed your work, save it as a pdf.

b) If you do not have a tablet with a stylus, but you do have access to a **printer**, print this pdf and write your answers to the exercises in the provided blank spaces. When you have completed your work, scan it with a printer or with a smartphone (in the latter case, you will need a **scanner app**, I personally use *Tiny scanner*)

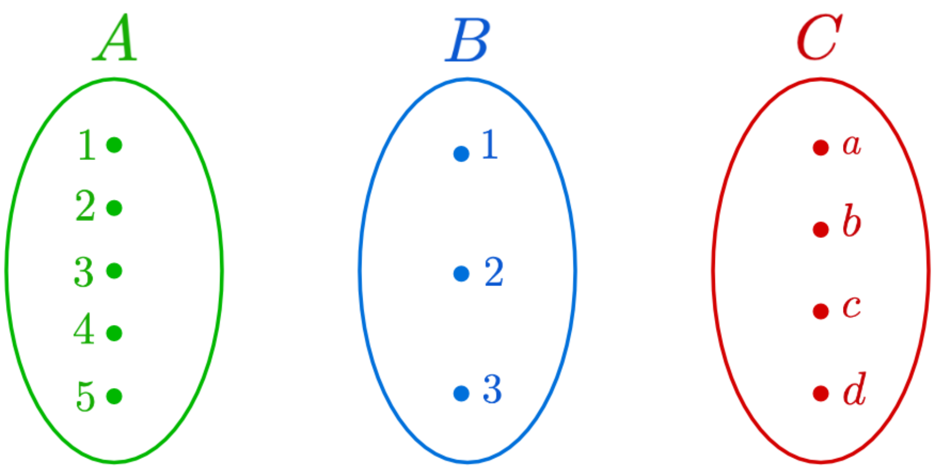
c) If you have neither a tablet, nor a printer, solve as usual these exercises on a separate sheet of paper. Please change paper when you change exercise. When you have completed your work, scan it with your smartphone (you will need a **scanner app**, I personally use *Tiny scanner*).

Exercise 1 (18 points) Consider the following sets

$$\begin{aligned}A &= \{1, 2, 3, 4, 5\}\\ B &= \{1, 2, 3\}\\ C &= \{a, b, c, d\}.\end{aligned}$$

Let R be a relation from A to B and S be a relation from B to C defined as follows:

$$\begin{aligned}R &= \{(1, 1), (2, 2), (2, 3), (4, 2), (5, 1)\}\\ S &= \{(1, a), (2, b), (2, d)\}\end{aligned}$$



(a) [6 points] Find R^{-1} (i.e. list all its elements).

(b) [6 points] Find $S \circ R$ (i.e. list all its elements).

(c) [6 points] Is $S \circ R$ a function?

☐ YES

☐ NO

Justify your answer:

Exercise 2 (41 points) Consider the following relation on $\mathbb{Z}$:

$$R = \{(a,b) \in \mathbb{Z}^2 : 4 \mid (a^2 - b^2)\}.$$

(a) [6 points] Which ordered pairs among the following ones belong to R? Select all that apply.

- ☐ (0,8)
- ☐ (9,11)
- ☐ (-8,5)
- ☐ (-11,-4)

(b) [7 points] Prove that R is reflexive on $\mathbb{Z}$.

(c) [7 points] Prove that R is symmetric.

(d) [7 points] Prove that R is transitive.

(e) [10 points] Let $a \in \mathbb{Z}$. Prove that $a \in \overline{0}$ if and only if a is even. (Recall that for proving a biconditional sentence you have to prove both implications).

(f) [4 points] Knowing that $\overline{1} = \{2k + 1 : k \in \mathbb{Z}\}$, describe $\mathbb{Z}/R$.

Exercise 3 (18 points) Consider the function

$$\begin{array}{rccc} \pi: & \mathbb{Z} & \to & \mathbb{Z}_{11} \\ & x & \mapsto & \overline{x} \end{array}$$

(a) [6 points] Show that π is <u>not</u> one-to-one.

(b) [6 points] Describe all the pre-images of $\overline{7} \in \mathbb{Z}_{11}$.

(c) [6 points] Prove that π is onto $\mathbb{Z}_{11}$.

Exercise 4 (18 points) Consider the function

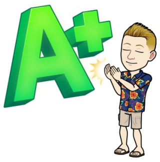
$$\begin{array}{rccc} g: & (-\infty, 1] & \to & \mathbb{R} \\ & x & \mapsto & 2+\sqrt{1-x} \end{array}$$

(a) [8 points] Prove that g is one-to-one.

(b) [10 points] Find the range of g and explain fully your answer.

Exercise 5 (15 points) Prove that for all $n \in \mathbb{N}$, $9 \mid (10^n - 1)$.

Exercise 6 (10 points)



Anna, I'll prove you now by induction that all your bridge students will get the same grade in Test 3! Check this out!

Proof. Let $n \in \mathbb{N}$ and let

$P(n) =$ "For any set S of n students, all the students in S will get the same grade in Test 3".

- Basis step: For $n = 1$, we have $P(1) =$"For any set S of 1 student, all the students in S will get the same grade in Test 3", which is trivially true, since if there is only one student in the set, then clearly all the students in that set will get the same grade.

- Inductive step: Let us assume that $P(n)$ is true. We will prove that $P(n+1)$ is true, i.e. that for any set S of $n+1$ students, all the students in S will get the same grade in Test 3.

  Let $S = \{x_1, \ldots, x_{n+1}\}$ be an arbitrary set with $n+1$ students. Then

$$S = \{\rlap{$\overbrace{}^{S'}$}x_1, \underbrace{x_2 \ldots, x_n, x_{n+1}}_{S''}\}$$

  where S' and S'' are two subsets of S with n students each. By inductive hypothesis all the students in S' will get the same grade g_1 and all the students in S'' will get the same grade g_2. Now, since the student $x_2 \in S' \cap S''$ can not get two different grades, we have $g_1 = g_2$. We conclude that all the students in S will get the same grade. Therefore $P(n+1)$ is true.

By induction we obtain that $P(n)$ is true for every n, which means that all the students will get the same grade in Test 3.

$\square$

Diego, I would love that, especially if they were all getting A+, but this just does not sound possible! You should be wrong somewhere in your proof... I'll ask to my bridge students to help me spotting your mistake!

Dear bridge student, could you help Anna to find the mistake in Diego's proof?