Bridge - MGF 3301 - Section 001

TEST 3 - Solution

04/22/2020

Instructions

This test contains 6 exercises. The total number of points is 120, but your grade will be the minimum between your score and 110 (you can get up to 10 bonus points).

In each proof you may use a result proved in class, in the homework or in previous exercises of this test. In this case just state clearly which result you are using, but there is no need of proving it again.

Neatness and clarity are important. You will lose credit if we cannot decipher your answer.

This exam is open-book and open-notes. You are not allow to discuss the exercises of this test with any other student.

When you have completed your work, please submit it by 11am on Gradescope.com, under the assignment *Test 3*. Remember that you have to submit one unique pdf.

For that as always you will have three options:

a) If you have a **tablet with a stylus**, write your answers to the exercises directly on this pdf, in the provided blank spaces. When you have completed your work, save it as a pdf.

b) If you do not have a tablet with a stylus, but you do have access to a **printer**, print this pdf and write your answers to the exercises in the provided blank spaces. When you have completed your work, scan it with a printer or with a smartphone (in the latter case, you will need a **scanner app**, I personally use *Tiny scanner*)

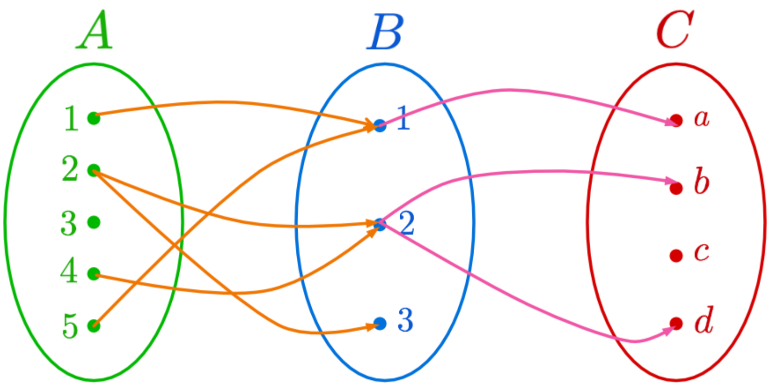
c) If you have neither a tablet, nor a printer, solve as usual these exercises on a separate sheet of paper. Please change paper when you change exercise. When you have completed your work, scan it with your smartphone (you will need a **scanner app**, I personally use *Tiny scanner*).

Exercise 1 (18 points) Consider the following sets

$$A = \{1, 2, 3, 4, 5\}$$
$$B = \{1, 2, 3\}$$
$$C = \{a, b, c, d\}.$$

Let R be a relation from A to B and S be a relation from B to C defined as follows:

$$R = \{(1, 1), (2, 2), (2, 3), (4, 2), (5, 1)\}$$
$$S = \{(1, a), (2, b), (2, d)\}$$



(a) [6 points] Find R^{-1} (i.e. list all its elements).

Solution

$R^{-1} = \{(1, 1), (2, 2), (3, 2), (2, 4), (1, 5)\}$

(b) [6 points] Find $S \circ R$ (i.e. list all its elements).

Solution

$S \circ R = \{(1, a), (2, b), (2, d), (4, b), (4, d), (5, a)\}$

(c) [6 points] Is $S \circ R$ a function?

- ☐ YES
- ■ NO

Justify your answer:

Solution

There are two possible different justifications:

1) $\text{Dom}(S \circ R) = \{1, 2, 4, 5\} \neq A$.
2) $(2, b), (2, d) \in S \circ R$, and $b \neq d$.

Exercise 2 (41 points) Consider the following relation on $\mathbb{Z}$:

$$R = \{(a, b) \in \mathbb{Z}^2 : 4 \mid (a^2 - b^2)\}.$$

(a) [6 points] Which ordered pairs among the following ones belong to R? Select all that apply.

- ■ (0,8)
- ■ (9,11)
- ☐ (-8,5)
- ☐ (-11,-4)

(b) [7 points] Prove that R is reflexive on $\mathbb{Z}$.

Solution

For all $a \in \mathbb{Z}$ we have that $a^2 - a^2 = 0$ and $4 \mid 0$. So $(a, a) \in R$, $\forall\, a \in \mathbb{Z}$, which implies that R is reflexive.

(c) [7 points] Prove that R is symmetric.

Solution

Let $a, b \in \mathbb{Z}$ such that $(a, b) \in R$, i.e. $4 \mid (a^2 - b^2)$. Then, by definition, there exists $k \in \mathbb{Z}$ such that $a^2 - b^2 = 4k$. Then $b^2 - a^2 = 4 \cdot (-k)$, which implies that $4 \mid (b^2 - a^2)$. Hence $(b, a) \in R$, which proves that R is symmetric.

(d) [7 points] Prove that R is transitive.

Solution

Let $a, b, c \in \mathbb{Z}$. Assume that $(a, b), (b, c) \in R$, i.e. $4 \mid (a^2 - b^2)$ and $4 \mid (b^2 - c^2)$. Then, by definition, there exist $k, h \in \mathbb{Z}$ such that $a^2 - b^2 = 4k$ and $b^2 - a^2 = 4h$. We have:

$$a^2 - c^2 = a^2 - b^2 + b^2 - c^2 = 4k + 4h = 4(k + h).$$

Therefore $4 \mid (a^2 - c^2)$, which implies $(a, c) \in R$, proving that R is transitive.

(e) [10 points] Let $a \in \mathbb{Z}$. Prove that $a \in \overline{0}$ if and only if a is even. (Recall that for proving a biconditional sentence you have to prove both implications).

Solution

Proof.

$\Rightarrow$) Let $a \in \mathbb{Z}$. Assume that $a \in \overline{0} = \{b \in \mathbb{Z} : (b, 0) \in R\}$. Then $(a, 0) \in R$, or equivalently $4 \mid a^2 - 0$. Hence there exists $k \in \mathbb{Z}$ such that $a^2 = 4k = 2 \cdot 2k$. We obtain that a^2 is even which implies that a is even (by a result proved in class).

$\Leftarrow$) Let $a \in \mathbb{Z}$. Assume that a is even, that is there exists $k \in \mathbb{Z}$ such that $a = 2k$. Then $a^2 = 4k^2$. Hence $4 \mid (a^2 - 0)$ which implies that $(a, 0) \in R$, i.e. $a \in \overline{0}$.

$\square$

(f) [4 points] Knowing that $\overline{1} = \{2k + 1 : k \in \mathbb{Z}\}$, describe $\mathbb{Z}/R$.

Solution

Since $\overline{0}$ and $\overline{1}$ consist respectively of all even and odd integers, the relation R has no other equivalence classes. As a consequence

$$\mathbb{Z}/R = \{\overline{0}, \overline{1}\}.$$

Exercise 3 (18 points) Consider the function

$$\begin{array}{rccc} \pi: & \mathbb{Z} & \to & \mathbb{Z}_{11} \\ & x & \mapsto & \overline{x} \end{array}$$

(a) [6 points] Show that π is <u>not</u> one-to-one.

Solution

We have:

$$\pi(1) = \overline{1} = \overline{1}$$
$$\pi(12) = \overline{12} = \overline{11 \cdot 1 + 1} = \overline{1}$$

So 1 and 12 are two different integers that are in the same equivalence class modulo 11 ($12 \equiv 1 \mod 11$). Therefore π is not one-to-one.

(b) [6 points] Describe all the pre-images of $\overline{7} \in \mathbb{Z}_{11}$.

Solution

The pre-images of $\overline{7} \in \mathbb{Z}_{11}$ are

$$\begin{aligned} \pi^{-1}(\overline{7}) &= \{x \in \mathbb{Z} : \pi(x) = \overline{7}\} = \\ &= \{x \in \mathbb{Z} : x = 11k + 7,\ k \in \mathbb{Z}\}. \end{aligned}$$

(c) [6 points] Prove that π is onto $\mathbb{Z}_{11}$.

Solution

We have to show that for every $y \in \mathbb{Z}_{11} = \{\overline{0}, \overline{1}, \ldots, \overline{10}\}$ there exists $x \in \mathbb{Z}$ such that $\pi(x) = y$. Let $y = \overline{r}$, with r an integer such that $0 \le r \le 10$. Then we have

$$\pi(r) = \overline{r} = y.$$

We conclude that π is onto $\mathbb{Z}_{11}$.

Exercise 4 (18 points) Consider the function

$$\begin{array}{rccc} g: & (-\infty,1] & \to & \mathbb{R} \\ & x & \mapsto & 2+\sqrt{1-x} \end{array}$$

(a) [8 points] Prove that g is one-to-one.

Solution

Let $x, y \in (-\infty, 1]$ and assume that $g(x) = g(y)$. Then we have

$$\begin{gathered} 2+\sqrt{1-x} = 2+\sqrt{1-y} \\ \Downarrow \\ \sqrt{1-x} = \sqrt{1-y} \\ \Downarrow \\ 1-x = 1-y \\ \Downarrow \\ x = y \end{gathered}$$

Therefore, by definition, g is one-to-one.

(b) [10 points] Find the range of g and explain fully your answer.

Solution

We will prove, by double inclusion, that $\mathrm{Rng}(g) = [2, \infty)$.

$\subseteq$) Let $y \in \mathrm{Rng}(g)$. Then there exists $x \in (-\infty, 1]$ such that $g(x) = y$, i.e.

$$y = 2+\sqrt{1-x}.$$

Note that $\sqrt{1-x} \geq 0$, $\forall x \in (-\infty, 1]$. So $y \geq 2$, or in other words $y \in [2, \infty)$.

$\supseteq$) Let $y \in [2, \infty)$. Then $y \geq 2$. Set $x = 1-(y-2)^2$. We have:

$$\begin{aligned} g(x) &= g(1-(y-2)^2) = 2+\sqrt{1-1+(y-2)^2} = \\ &= 2+\sqrt{(y-2)^2} = 2+|y-2| \overset{y-2\geq 0}{=} 2+y-2 = y. \end{aligned}$$

We conclude that $y \in \mathrm{Rng}(g)$.

Exercise 5 (15 points) Prove that for all $n \in \mathbb{N}$, $9 \mid (10^n - 1)$.

Solution

Proof:

Let $P(n) =$ "$9 \mid (10^n - 1)$". We will prove the statement by induction.

- Basis step: For $n = 1$, we have $P(1) =$ "$9 \mid (10^1 - 1)$" $=$ "$9 \mid 9$", so $P(1)$ is true.

- Inductive step: Let us assume that $P(n)$ is true, i.e. that

$$9 \mid (10^n - 1).$$

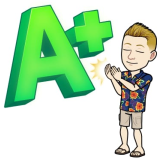
  By definition there exists $k \in \mathbb{Z}$ such that $10^n - 1 = 9k$. We want to show that $P(n+1) =$ "$9 \mid (10^{n+1} - 1)$" is also true. We have:

$$\begin{aligned} 10^{n+1} - 1 &= \underbrace{10^n}_{9k+1} \cdot 10 - 1 = (9k+1) \cdot 10 - 1 = \\ &= 90k + 10 - 1 = 9 \cdot (10k + 1). \end{aligned}$$

  Therefore $9 \mid (10^{n+1} - 1)$ and $P(n+1)$ is also true.

In conclusion, by induction, $\forall\, n \in \mathbb{N}$, $9 \mid (10^n - 1)$.

Exercise 6 (10 points)



Anna, I'll prove you now by induction that all your bridge students will get the same grade in Test 3! Check this out!

Proof. Let $n \in \mathbb{N}$ and let

$P(n) =$ "For any set S of n students, all the students in S will get the same grade in Test 3".

- Basis step: For $n = 1$, we have $P(1) =$"For any set S of 1 student, all the students in S will get the same grade in Test 3", which is trivially true, since if there is only one student in the set, then clearly all the students in that set will get the same grade.
- Inductive step: Let us assume that $P(n)$ is true. We will prove that $P(n+1)$ is true, i.e. that for any set S of $n+1$ students, all the students in S will get the same grade in Test 3.

  Let $S = \{x_1, \dots, x_{n+1}\}$ be an arbitrary set with $n+1$ students. Then

  $$S = \{\rlap{$\overbrace{}^{S'}$}x_1, \underbrace{x_2 \dots, x_n, x_{n+1}}_{S''}\}$$

  where S' and S'' are two subsets of S with n students each. By inductive hypothesis all the students in S' will get the same grade g_1 and all the students in S'' will get the same grade g_2. Now, since the student $x_2 \in S' \cap S''$ can not get two different grades, we have $g_1 = g_2$. We conclude that all the students in S will get the same grade. Therefore $P(n+1)$ is true.

By induction we obtain that $P(n)$ is true for every n, which means that all the students will get the same grade in Test 3.

$\square$

Diego, I would love that, especially if they were all getting A+, but this just does not sound possible! You should be wrong somewhere in your proof... I'll ask to my bridge students to help me spotting your mistake!

Dear bridge student, could you help Anna to find the mistake in Diego's proof?

Solution

The problem in the reasoning of Diego is that an element x_2 can not be find in the intersection of S' and S'' for every n. In particular, for moving from $P(1)$ to $P(2)$, we would have to consider a set S with two students:

$$S = \{x_1, x_2\}.$$

For applying the inductive hypothesis we now would need to work with subsets of one student. But if we define as in Diego's proof

$$S' = \{x_1\} \quad \text{and} \quad S'' = \{x_2\},$$

we would get $S' \cap S'' = \varnothing$, so the last part of Diego's reasoning fails and there is no way of proving that the grades of x_1 and x_2 are the same.