Bridge - MGF 3301 - Section 001

Quiz 4 - Solution

02/26/2020

Instructions: The total number of points for this quiz is 14. However, your final score will be the minimum between the total number of your points and 11. Calculators are not allowed (and actually not needed).

EXERCISE 1
(7 points)

Recall the following definition from the homework:

Definition

Given two integers a and b we say that a **divides** b, and we write $a|b$, if there exists an integer k such that

$$b = ka.$$

Moreover, we write $a \nmid b$ if a does not divide b.

Prove by contrapositive the following claim (please, write down the contrapositive of the statement first):

Claim: *Let a and b in $\mathbb{Z}$. If $5 \nmid ab$, then $5 \nmid a$ and $5 \nmid b$.*

Solution

Contrapositive: Let a and b in $\mathbb{Z}$. If $5 \mid a$ or $5 \mid b$, then $5 \mid ab$.

Proof. We will prove the claim by proving its contrapositive. Let $a, b \in \mathbb{Z}$. Assume that $5 \mid a$ or $5 \mid b$. We have to consider then two separate cases:

Case 1: If $5 \mid a$ then, by definition, there exists $k \in \mathbb{Z}$ such that $a = 5k$. Then $ab = (5k)b = 5(kb) = 5h$, with $h = kb \in \mathbb{Z}$. Then $5 \mid ab$.

Case 2: If $5 \mid b$ then, by definition, there exists $k \in \mathbb{Z}$ such that $b = 5k$. Then $ab = a(5k) = 5(ak) = 5h$, with $h = ak \in \mathbb{Z}$. Then $5 \mid ab$.

So in any case $5 \mid ab$. $\square$

Exercise 2
(7 points)

Prove by contradiction the following claim and highlight what is the contradiction (i.e. identify the proposition Q such that you have $Q \wedge (\sim Q)$). Note that you may use previous results proved in class. In case state them.

Claim: *For all a and b in $\mathbb{Z}$, $a^2 - 4b - 2 \neq 0$.*

Solution

Proof. By contradiction, assume that there exist $a, b \in \mathbb{Z}$ such that $a^2 - 4b - 2 = 0$. Then $a^2 = 4b + 2 = 2(b+1)$, which means that a^2 is an even integer. Because of a previous result seen in class ("*n is even $\Leftrightarrow$ n^2 is even*"), this implies that a is also even. Then, by definition, there exists $k \in \mathbb{Z}$ such that $a = 2k$. So we obtain:

$$(2k)^2 - 4b - 2 = 0 \Leftrightarrow 4k^2 - 4b - 2 = 0 \Leftrightarrow 2k^2 = 2b + 1.$$

So $2k^2$ is odd (since in the last identity we see that it can be written in the form $2h + 1$ for some $h \in \mathbb{Z}$), but it is also even (since it can be written in the form 2ℓ for some $\ell \in \mathbb{Z}$). This is a contradiction, since an integer can not be even and odd at the same time. Therefore for all a and b in $\mathbb{Z}$, $a^2 - 4b - 2 \neq 0$.

Contradiction: "$2k^2$ is odd" and "$2k^2$ is even"

$\square$