INVERSE FUNCTIONS

(Sec. 4.4)

Let $f: A \to B$ be a function.

Let us consider the inverse relation

$$f^{-1} = \{ (y,x) : \underbrace{(x,y) \in f}_{f(x)=y \Rightarrow y \in Rng(f)} \} \subseteq Rng(f) \times A \subseteq B \times A$$

We have noticed that f^{-1} is not always a function.

Ex: $f: \mathbb{R} \to \mathbb{R}$, $f(x) = x^2$

$$f^{-1} = \{ (y,x) : (x,y) \in f \} = \{ (x^2, x) : x \in \mathbb{R} \}$$

is a relation which is not a function

$(1,1) \in f^{-1}$, $(1,-1) \in f^{-1}$, but $1 \neq -1$

Question: When is f^{-1} a function?

Theorem: Let $f: A \to B$ be a function. Then $f^{-1} \subseteq Rng(f) \times A$ is a function $\Leftrightarrow$ f is one-to-one. In this case $f^{-1}: Rng(f) \to A$ is a one-to-one function with range equal to A.

Proof:

Let $f: A \to B$ be a function.

$\Leftarrow$) We have to to prove that if f is one-to-one then f^{-1} is a function from $Rng(f)$ to A.

1) $Dom(f^{-1}) = Rng(f)$

2) If $(x,y_1), (x,y_2) \in f^{-1} \Rightarrow y_1 = y_2$

1) $\mathrm{Dom}(f^{-1}) = \mathrm{Rng}(f)$.

$y \in \mathrm{Dom}\, f^{-1} \iff \exists\, x \in A$ such that $(y,x) \in f^{-1}$

$\iff \exists\, x \in A$ such that $\underbrace{(x,y) \in f}_{f(x)=y} \implies$

$\iff y \in \mathrm{Rng}(f)$

2) Let $x \in \mathrm{Rng}(f)$, $y_1, y_2 \in A$ such that $(x,y_1), (x,y_2) \in f^{-1}$

$\implies y_1 = y_2$.

$(x, y_1), (x, y_2) \in f^{-1} \implies (y_1, x), (y_2, x) \in f \implies$

$\implies f(y_1) = x = f(y_2) \xRightarrow[\text{f is one-to-one}]{} y_1 = y_2$.

$\Rightarrow$) Assume that f^{-1} is a function from $\mathrm{Rng}(f)$ to A.

Let $x, y \in A$ s.t. $f(x) = f(y) = z$, with $z \in \mathrm{Rng}(f)$.

$\implies (x,z), (y,z) \in f \implies (z,x), (z,y) \in f^{-1} \implies$

$\xRightarrow[\text{$f^{-1}$ a function}]{} x = y$. Therefore f is one-to-one

Assume now that $f^{-1} : \mathrm{Rng}(f) \to A$ is a function.

Let us show that f^{-1} is one-to-one and onto A

- f^{-1} one-to-one

  Let $x, y \in \mathrm{Rng}(f)$ such that $f^{-1}(x) = f^{-1}(y) = z$, with $z \in A$. Then $(x,z), (y,z) \in f^{-1} \implies (z,x), (z,y) \in f$

  $\xRightarrow[\text{$f$ is a function}]{} x = y$. Therefore f^{-1} is one-to-one.

- f^{-1} is onto A.

Let $x \in A$. Since $A = \mathrm{Dom}(f)$, $\exists\, y \in \mathrm{Rng}(f)$ such that $(x,y) \in f \Rightarrow \exists\, y \in \mathrm{Rng}(f)$ such that $(y,x) \in f^{-1}$ $\Rightarrow x \in \mathrm{Rng}(f^{-1})$. Therefore f^{-1} is onto A.

$\square$

You have defined in calculus also the inverse of functions which are not one-to-one (ex. trigonometric functions).
In this case you restrict the domain in a way that the function becomes one to one.

Def: Let $f : A \to B$ be a function and let $D \subseteq A$.
The <u>restriction</u> of f to D is the function

$$f\big|_D := \{ (x,y) : y = f(x),\ x \in D \}$$

<u>Example</u>:
$$\sin : \mathbb{R} \to \mathbb{R}$$
$$x \mapsto \sin(x)$$

$\sin(0) = 0 = \sin(2\pi) \Rightarrow \sin$ is not one-to-one

Nevertheless, if we restrict the domain of sin to $\left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$, the function

$$\sin\big|_{\left[-\frac{\pi}{2}, \frac{\pi}{2}\right]} : \left[-\frac{\pi}{2}, \frac{\pi}{2}\right] \to \mathbb{R}$$
$$x \mapsto \sin(x)$$

becomes one-to-one and has range $[-1,1]$.
Therefore the inverse relation

$$\sin^{-1} : [-1,1] \to \left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$$

is a function which is defined as $\forall\, x \in [-1,1],\ y \in \left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$

$$(x,y) \in \sin^{-1} \iff (y,x) \in \sin$$
$$\sin^{-1}(x) = y \iff \sin(y) = x$$