FUNCTIONS
(Sec. 4.1, 4.2, 4.3)

Recall: $f: \underset{\mathrm{Dom}(f)}{A} \to B$ codomain Always $\mathrm{Rng}(f) \subseteq B$

$f \subseteq A \times B$ such that:

1) $\mathrm{Dom}(f) = \{x \in A : \exists\, y \in B \text{ such that } (x,y) \in f\}$
2) $\forall x \in A,\ \forall y, z \in B$ if $(x,y), (x,z) \in f$ then $y = z$.

$(x,y) \in f \iff f(x) = y$

Example: $A = B = \mathbb{R}$

$f = \{(x, \overset{y}{x^2}) : x \in \mathbb{R}\} \subseteq \mathbb{R} \times \mathbb{R}$

$(1,0) \notin f$ because $1^2 \neq 0$.

$f: \mathbb{R} \to \mathbb{R}$, $x \mapsto x^2$, $\underbrace{f(x)}_{= y} = x^2$

codomain $\mathbb{R}$

$\mathrm{Rng}(f) = [0, \infty) \subsetneq \mathbb{R}$

What does it mean $f = g$, where f and g are functions?

$f = g$ as sets ($f \subseteq g$ and $g \subseteq f$).
(x,y)

Theorem: Two functions f and g are equal if and only if:

1) $\mathrm{Dom}(f) = \mathrm{Dom}(g)$.
2) $\forall x \in \mathrm{Dom}(f)$, $f(x) = g(x)$.

Theorem 4.1.1

$f: \mathbb{R} \to \mathbb{R}$, $g: \mathbb{R} \to \mathbb{R}$

$f(x) = x - 1 \longrightarrow \mathrm{Dom}(f) = \mathbb{R}$

$g(x) = \dfrac{x(x-1)}{x} \longrightarrow \mathrm{Dom}(g) = \mathbb{R} \setminus \{0\}$

$\Rightarrow f \neq g$ (but $f(x) = g(x)$ $\forall x \in \mathbb{R} \setminus \{0\}$)

Typical examples of functions

- **Identity function** (Identity relation)

Let A be any set, $\quad I_A : A \longrightarrow A$

$\qquad\qquad\qquad\qquad x \longmapsto x$

$$I_A = \{(x,x) : x \in A\}$$

- **Inclusion function**

Let A, B be two sets such that $A \subseteq B$.

$$i : A \longrightarrow B$$
$$x \longmapsto x$$

same functions (same range). Different codomain

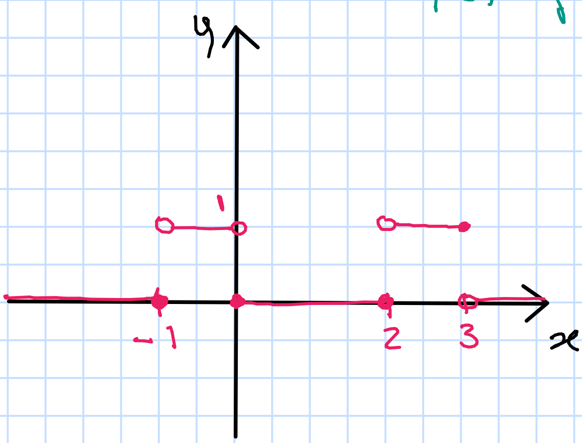
- **Characteristic function**

Let $A \subseteq U$

$$\chi_A : U \longrightarrow \{0,1\}$$
$$x \longmapsto \begin{cases} 1, & \text{if } x \in A \\ 0, & \text{if } x \notin A \end{cases}$$

Example : $A = (-1,0) \cup (2,3] \subseteq \mathbb{R} = U$

$$\chi_A : \mathbb{R} \longrightarrow \{0,1\}$$
$$x \longmapsto \begin{cases} 1, & \text{if } x \in A \\ 0, & \text{if } x \notin A \end{cases}$$



Canonical map : Let R be an equivalence relation on a set X.

$$\pi : X \longrightarrow X/R$$
$$x \longmapsto \bar{x}$$

example : $X = \mathbb{Z}$, $R = $ " $\equiv_5$ "

$$\pi : \mathbb{Z} \longrightarrow \mathbb{Z}_5 = \mathbb{Z}/R = \{\bar{0}, \bar{1}, \bar{2}, \bar{3}, \bar{4}\}$$
$$x \longmapsto \bar{x}$$
$$17 \longmapsto \overline{17} = \bar{2}$$

- Infinite sequence

A function with domain $\mathbb{N}$

$$f : \mathbb{N} \longrightarrow X$$
$$n \longmapsto x_n$$

Def: Let $f : A \to B$ be a function ($f \subseteq A \times B$)

The inverse of f is the relation from B to A:

$$\underset{R^{-1}}{f^{-1}} = \{(y, x) : (x, y) \in \underset{R}{f}\}$$

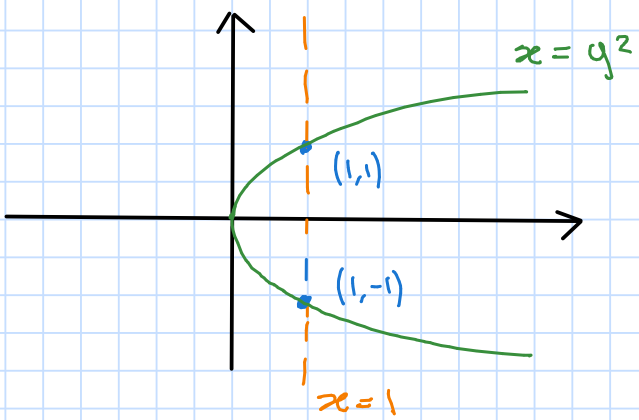
Warning: f^{-1} is not a function in general.

Example: $f : \mathbb{R} \longrightarrow \mathbb{R}$
$x \longmapsto x^2$

$$= \{(x, y) \in \mathbb{R} \times \mathbb{R} : y = x^2\} \Rightarrow$$

$$\Rightarrow f^{-1} = \{(y, x) \in \mathbb{R} \times \mathbb{R} : y = x^2\} =$$

$$= \{(x, y) \in \mathbb{R} \times \mathbb{R} : x = y^2\}$$



$(0,0) \in f^{-1}$

$1 \neq -1 \begin{cases} (\overset{x}{1}, \overset{y}{1}) \in f^{-1} \\ (\underset{x}{1}, \underset{z}{-1}) \in f^{-1} \end{cases}$

So f^{-1} is not a function

Def: Let $f: A \to B$ and $g: B \to C$ be functions ($f \subseteq A\times B$, $g \subseteq B \times C$).
The composite of f and g is the relation from A to C

$$\underset{S}{g} \circ \underset{R}{f} = \{(x,z) \in A\times C : \exists\, y \in B \text{ such that } (x,y) \in f \text{ and } (y,z) \in g\}$$

Example: $f: \mathbb{R} \to \mathbb{R}$, $x \mapsto 2x+1$ $\quad f = \{(x,y) \in \mathbb{R}\times\mathbb{R} : y = 2x+1\}$

$g: \mathbb{R} \to \mathbb{R}$, $x \mapsto x^2-3$ $\quad g = \{(x,y) \in \mathbb{R}\times\mathbb{R} : y = x^2-3\}$

$$g\circ f = \{(x,z) \in \mathbb{R}\times\mathbb{R} : \exists\, y \in \mathbb{R} \text{ such that } \underbrace{(x,y) \in f}_{y=2x+1} \text{ and } \underbrace{(y,z) \in g}_{z=y^2-3}\} =$$

$$= \{(x,z) \in \mathbb{R}\times\mathbb{R} : z = y^2-3 \text{ and } y = 2x+1\} =$$

$$= \{(x,z) \in \mathbb{R}\times\mathbb{R} : z = (2x+1)^2-3\}$$

$g\circ f: \mathbb{R} \to \mathbb{R}$, $x \mapsto (2x+1)^2 - 3$

$$(g\circ f)(x) = g(f(x)) = g(2x+1) = (2x+1)^2 - 3$$

Proposition: If $f: A \to B$ and $g: B \to C$ are functions then $g\circ f$ is function from A to C.

Proof:

- $\mathrm{Dom}(g\circ f) = A$
- $\forall x \in A$, $\forall z_1, z_2 \in C$, if $(x,z_1), (x,z_2) \in g\circ f$ $\Rightarrow z_1 = z_2$.

Theorem 4.2.1

<u>Warning</u>: $f\circ g \neq g\circ f$ in general.

ONE-TO-ONE and ONTO FUNCTIONS

<u>Def</u>: Let $f: A \to B$ be a function.
We say that f is one-to-one (or injective) if
① $\forall\, x,y \in A$, $x \neq y \Rightarrow f(x) \neq f(y)$
② $\forall\, x,y \in A$, $f(x) = f(y) \Rightarrow x = y$

- <u>To disprove that f is injective</u> use ①: find $x,y \in A$ with $x \neq y$ such that $f(x) = f(y)$
- <u>To prove that f is injective</u> use ②. Take $x,y \in A$ and assume $f(x) = f(y)$. Then conclude that $x = y$

<u>Examples</u>: 1) $f: \mathbb{R} \to \mathbb{R}$

$$x \longmapsto \frac{1}{x^2+1}$$

f is <u>not</u> injective because $f(1) = \frac{1}{2} = f(-1)$.

2) $f: [0,\infty) \to [0,\infty)$

$$x \longmapsto x^2$$

f is injective.

<u>Proof</u>: Let $x,y \in \mathbb{R}$ s.t. $f(x) = f(y) \Rightarrow$

$\Rightarrow x^2 = y^2 \Rightarrow \sqrt{x^2} = \sqrt{y^2} \Rightarrow$

$\Rightarrow |x| = |y| \underset{x,y \geq 0}{\Longrightarrow} x = y$.

Def: A function $f : A \to B$ is surjective if and only if $Rng(f) = B \iff B \subseteq Rng(f) \iff$

$Rng(f) \subseteq B$

$\iff \forall\, y \in B,\ \underbrace{\exists\, x \in A \text{ s.t. } f(x) = y}_{\text{y is the image of some elements in A.}}$.

Examples: 1) $f : [0,\infty) \to [0,\infty)$

$x \mapsto x^2$

$f(\sqrt{3}) = (\sqrt{3})^2 = 3$.

<u>f is surjective</u> (or onto $[0,\infty)$).

Let's prove that $\forall\, y \in [0,\infty),\ \exists\, x \in [0,\infty)$ s.t. $f(x) = y$.

Let $y \in [0,\infty)$ and set $x = \sqrt{y}$. We have $f(x) = (\sqrt{y})^2 = y$.

$x^2 = y$

$\Downarrow\ y \in [0,\infty)$

$x = \sqrt{y} \leftarrow$

or

$\boxed{x = -\sqrt{y}} \notin [0,\infty)$

2) $f : \mathbb{N} \to \mathbb{N}$

$n \mapsto n+5$

f is not surjective.

Let $y = 1 \in \mathbb{N}$.

We want to prove that $\forall\, x \in \mathbb{N},\ f(x) \neq 1$

Assume to the contrary that $\exists\, x \in \mathbb{N}$ s.t. $f(x) = 1 \Rightarrow x + 5 = 1 \Rightarrow x = -4$

$x > 0$, $-4 < 0$

because $x \in \mathbb{N}$.

3) $f: \mathbb{R} \to \mathbb{R}$

$x \mapsto \frac{1}{x^2+1} > 0$

Take $y=0$. Does there exist $x \in \mathbb{R}$ s.t. $f(x)=0$?

$\frac{1}{x^2+1} = 0 \Rightarrow 1 = 0$ ↯ So f is not onto $\mathbb{R}$

multiply by $x^2+1 \neq 0$

4) $g: \mathbb{R} \to (0,1]$

$x \mapsto \frac{1}{x^2+1}$

$y = \frac{1}{x^2+1} \Rightarrow (x^2+1)y = 1 \Rightarrow$

$\underset{y\neq 0}{\Rightarrow} x^2+1 = \frac{1}{y} \Rightarrow$

$x^2 = \frac{1}{y} - 1 \geq 0 \Rightarrow x = \sqrt{\frac{1}{y}-1}$

This is now surjective.

Let $y \in (0,1]$. We want to show that $\exists x \in \mathbb{R}$ s.t. $f(x) = y$

Let $y \in (0,1]$ and set $x = \sqrt{\frac{1}{y}-1}$. Then we have

$$g(x) = \frac{1}{\left(\sqrt{\frac{1}{y}-1}\right)^2+1} = \frac{1}{\frac{1}{y}-1+1} = \frac{1}{\frac{1}{y}} = y.$$