RELATIONS (Section 3.1)

<u>Def</u>: Let A, B be sets. A relation R from A to B is a subset of $A\times B$:

$$R \subseteq A\times B.$$

The <u>domain</u> of R is:

$$\mathrm{Dom}(R) := \{\underline{a\in A} : \exists\, b\in B \text{ s.t. } (a,b)\in R\} \subseteq A$$

The <u>range</u> of R is

$$\mathrm{Rng}(R) := \{\underline{b\in B} : \exists\, a\in A \text{ s.t. } (a,b)\in R\} \subseteq B$$

<u>Remark</u>:
- The domain is the set of all <u>first</u> coordinates of the ordered pairs in R
- The range is the set of all <u>second</u> coordinates of the ordered pairs in R.

<u>Example</u>:

$A = \{1, 2, 3, 4, 5\}$

$B = \{a, b, c, d\}$

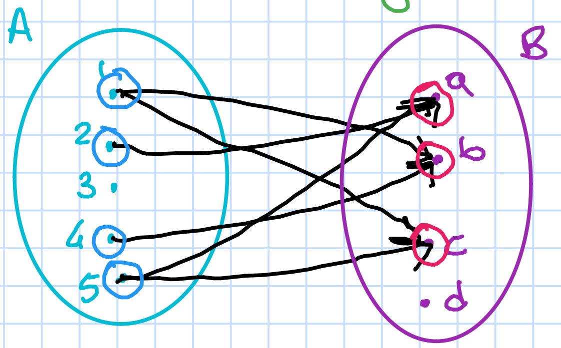
$R = \{(1,b), (1,c), (2,a), (4,b), (5,a), (5,c)\}$

$\mathrm{Dom}(R) = \{1, 2, 4, 5\}$

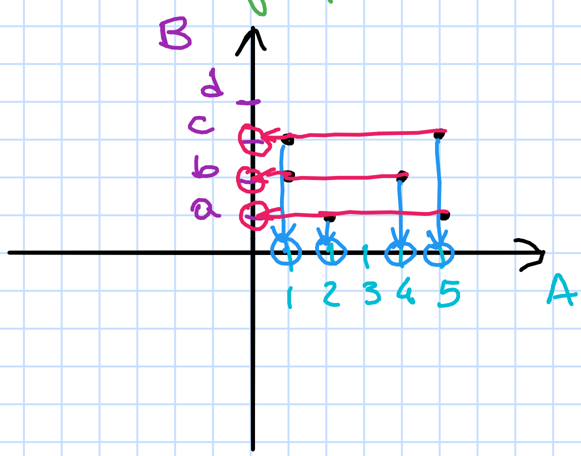
$\mathrm{Rng}(R) = \{a, b, c\}$

<u>Geometrically</u>

Arrow diagram



Graph



Def: If $R \subseteq A\times B$ is a relation from A to B then the <u>inverse of R</u> is the relation from B to A defined as:

$$R^{-1} = \{(b,a) : (a,b)\in R\} \subseteq B\times A$$

Example: $A = \{1,2,3,4,5\}$

$B = \{a,b,c,d\}$

$R = \{(1,b),(1,c),(2,a),(4,b),(5,a),(5,c)\} \subseteq A\times B$

$R^{-1} = \{(b,1),(c,1),(a,2),(b,4),(a,5),(c,5)\} \subseteq B\times A$

Example: $R = \{(x,y)\in \mathbb{R}\times\mathbb{R} : \underbrace{x^2+y^2\le 16}_{\text{relation between } x \text{ and } y}\}$

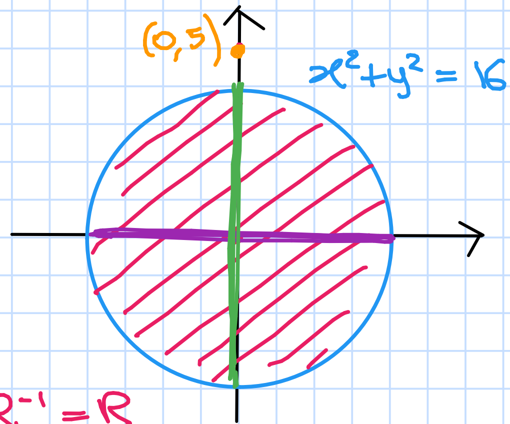
$(0,5)\notin R$ because $0^2+5^2 = 25 > 16$

$x \sim_R y \iff x^2+y^2 \le 16$.

Geometrically

$Dom(R) = [-4,4]$

$Rng(R) = [-4,4]$



In this example $R^{-1} = R$ (because the relation is symmetric).

Theorem: Let R be relation from A to B.

(1) $Dom(R^{-1}) = Rng(R) \subseteq B$

(2) $Rng(R^{-1}) = Dom(R) \subseteq A$

Proof $Dom(R^{-1}) \subseteq Rng(R)$ and $Rng(R) \subseteq Dom(R^{-1})$

(1) $b\in Dom(R^{-1}) \iff \exists\, a\in A$ s.t. $(b,a)\in R^{-1}$

$\iff \exists\, a\in A$ s.t. $(a,b)\in R \iff b\in Rng(R)$

Def: Let R be a relation from A to B and let S be a relation from B to C.

The <u>composite</u> of R and S is:

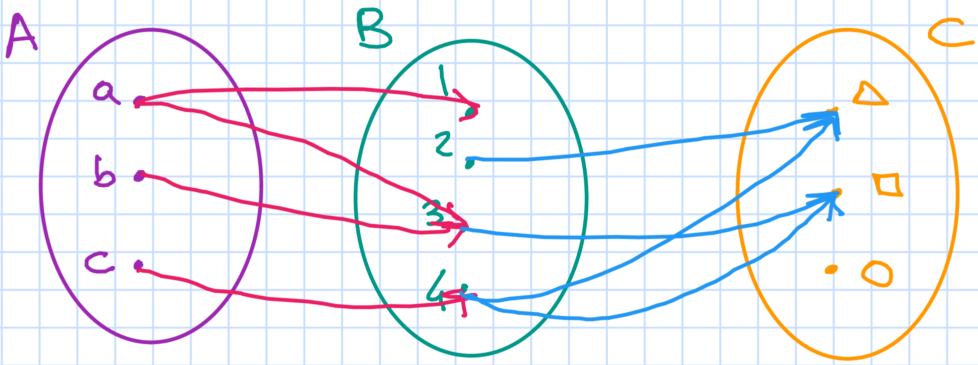
$$S \circ R := \{(a,c) \in A \times C : \exists\, b \in B \text{ s.t. } (a,b) \in R \text{ and } (b,c) \in S\} \subseteq A \times C.$$

intermediate between a and c

Example: $A = \{a, b, c\}$ $\quad B = \{1,2,3,4\}$ $\quad C = \{\triangle, \square, \circ\}$

$R = \{(a,1), (a,3), (b,3), (c,4)\} \subseteq A \times B$

$S = \{(2,\triangle), (3,\square), (4,\triangle), (4,\square)\} \subseteq B \times C.$



$$S \circ R = \{(a,\square), (b,\square), (c,\triangle), (c,\square)\}$$

first